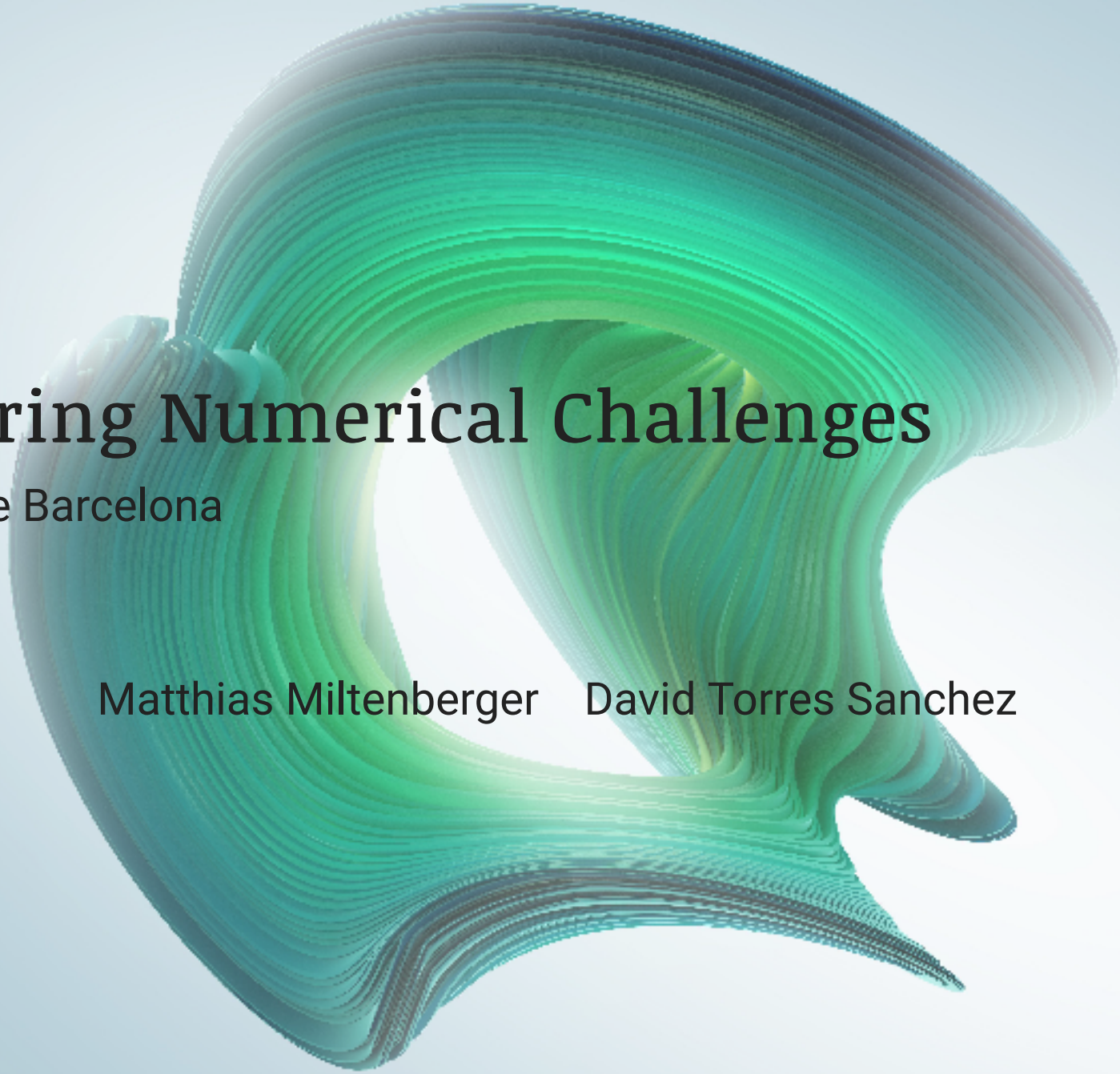


Mastering Numerical Challenges

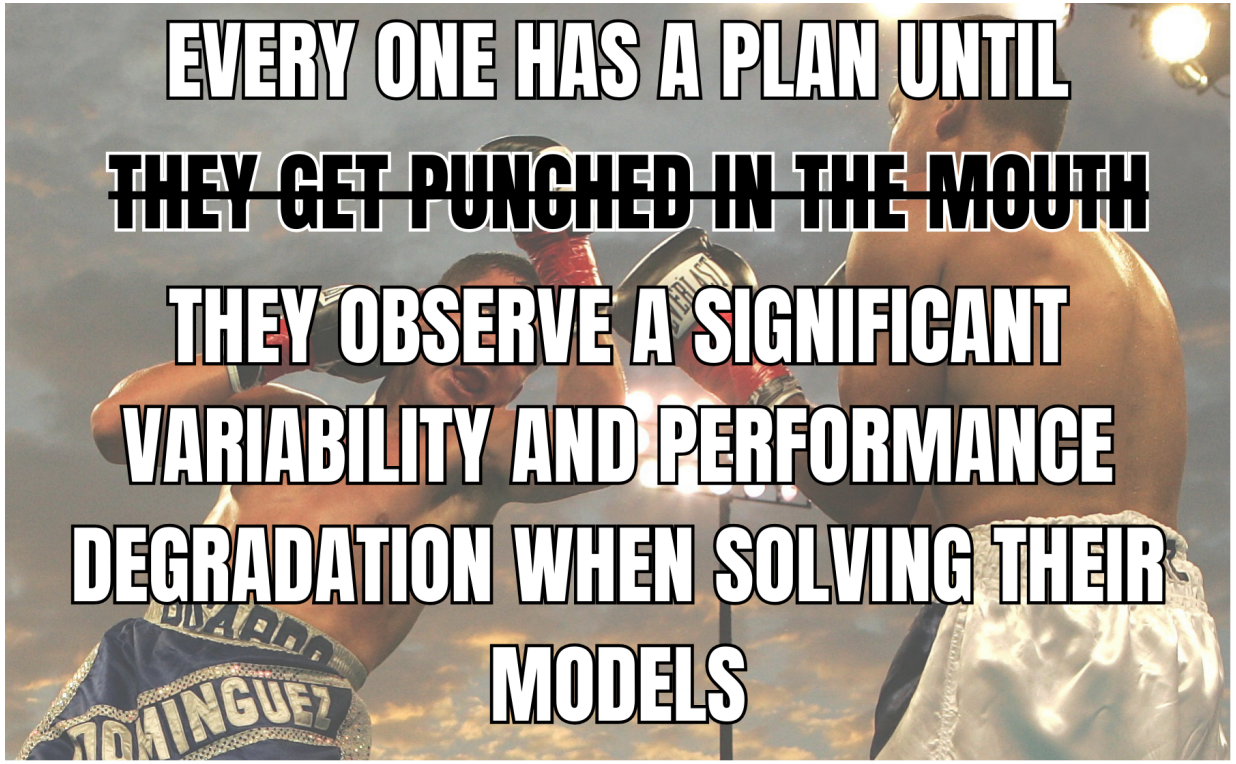
Gurobi Live Barcelona

Matthias Miltenberger David Torres Sanchez

GUROBI LiVE!

Introduction

- OR practitioners have to deal with unexpected, counter-intuitive results



Introduction

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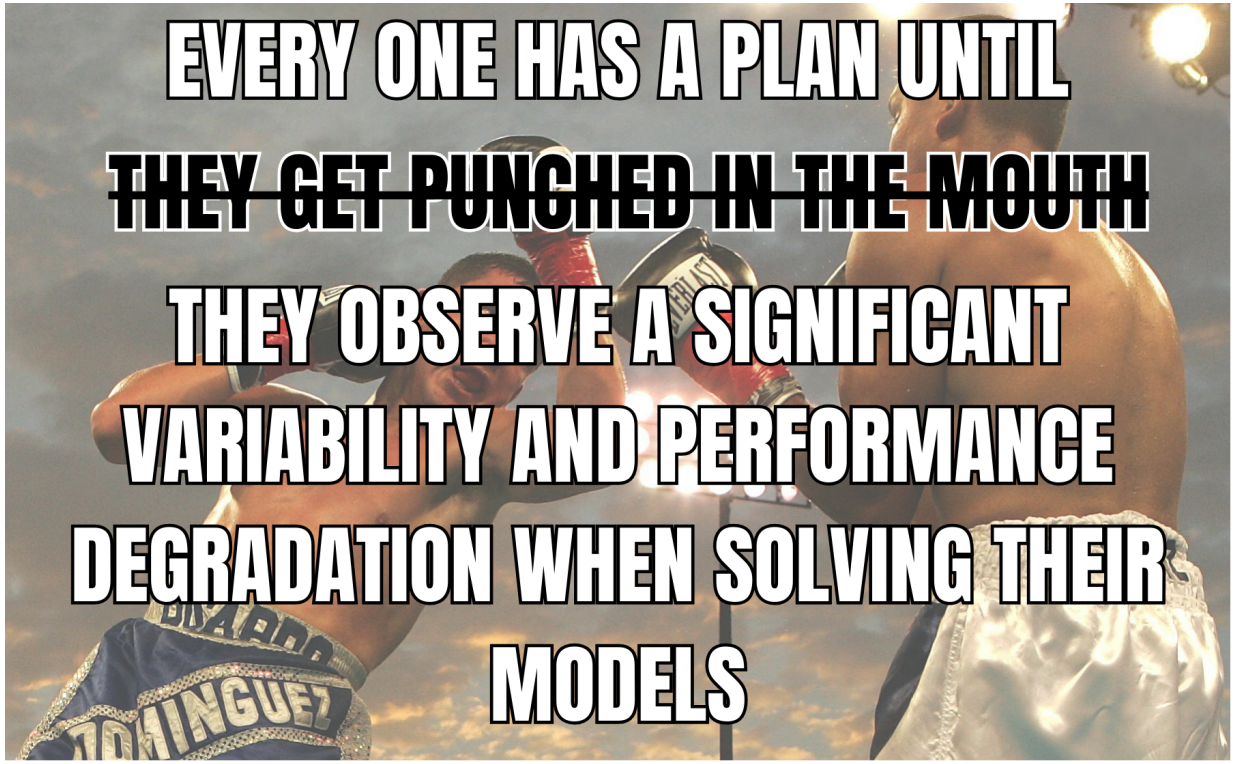
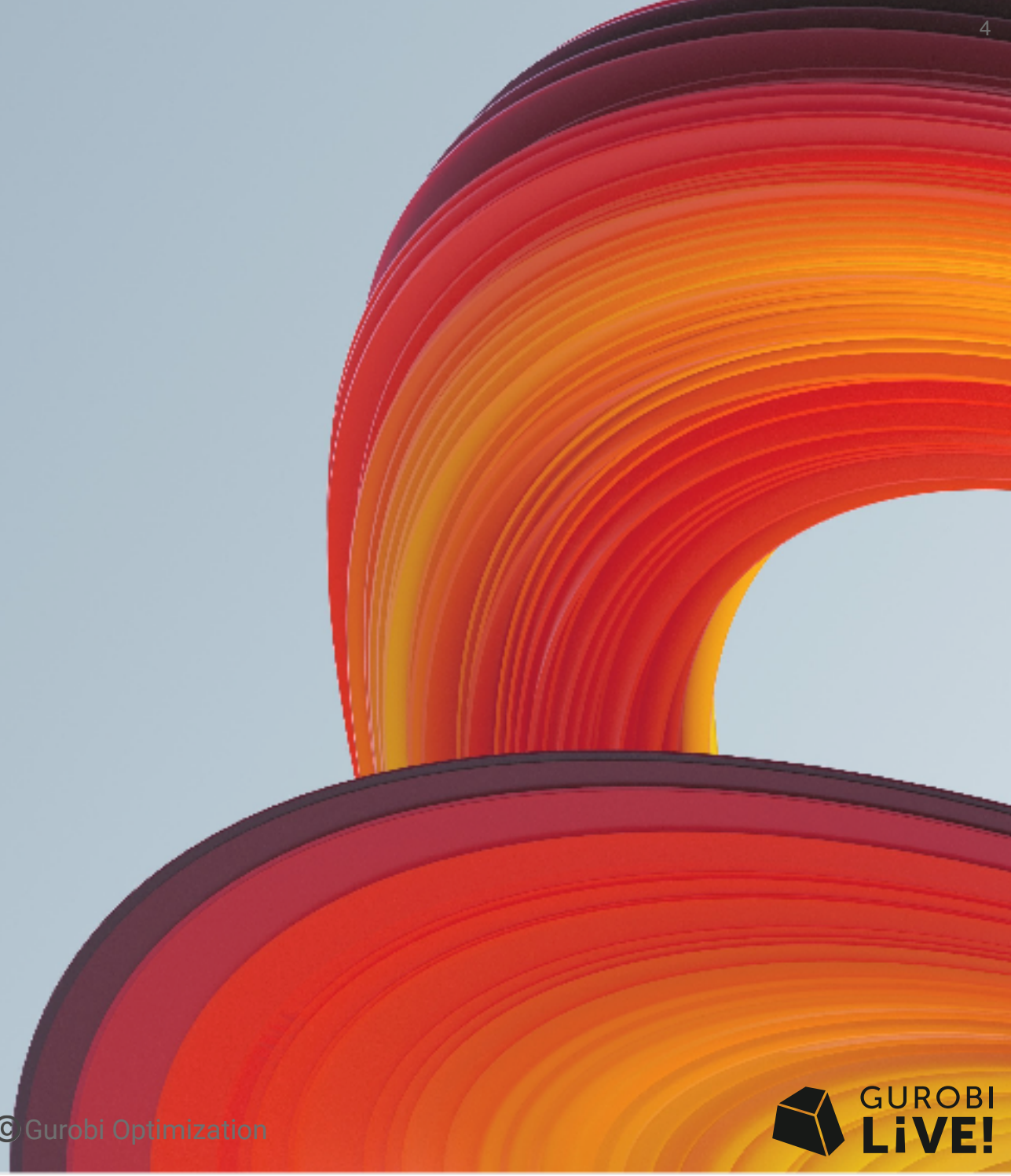


- We need an organized approach to deal with the unexpected

Agenda

1. **Concept**
   - Condition number
   - Why is this relevant?
   - Floating point numbers
2. **Practical recommendations**
   - Analyzing
   - Parameters
   - Summary
3. **Ill-Conditioning Explainer**



Condition Number
Definition

From

$$B(x + \delta x) = b + \delta b \rightarrow \delta x = B^{-1}\delta b$$

Cauchy-Schwartz inequalities:

$$\|\delta x\| \leq \|B^{-1}\|\|\delta b\|, \|b\| \leq \|B\|\|x\|$$

Combine:

$$\frac{\|\delta x\|}{\|x\|} \leq \underbrace{\|B\|\|B^{-1}\|}_{\kappa} \frac{\|\delta b\|}{\|b\|}$$

Condition Number
Definition

Definition of condition number κ of a square matrix B for solving $Bx = b$

$\kappa(B) = \|B\| \cdot \|B^{-1}\|$ provides upper bound on input error amplification in the solution of a linear system $Bx = b$. It is the normed reciprocal of the distance to singularity.

- Effect of the input error δb on the output error δx:

$$\frac{\|\delta x\|}{\|x\|} \leq \kappa(B) \cdot \frac{\|\delta b\|}{\|b\|}$$

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- Effect of the input error δb on the output error δx:

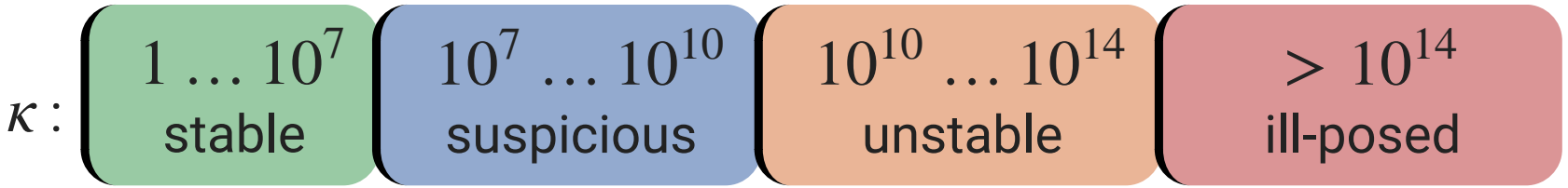
$$\frac{\|\delta x\|}{\|x\|} \leq \kappa(B) \cdot \frac{\|\delta b\|}{\|b\|}$$

- Error of 10^{-16} in the input b and $\kappa \approx 10^{10}$ can cause error up to 10^{-6} in x

Double precision

Default feasibility tolerance

Condition Number
Value Range



Note

Condition number always represents an **upper bound** or **worst-case scenario**!

Condition Number
Interpretations

Linear Algebra Interpretation

- How linearly dependent hyperplanes are.
  - Rows are linearly dependent if and only if B is singular, or columns are linearly dependent.
- How close B is to being singular

Condition Number
Interpretations

Linear Algebra Interpretation

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- How close B is to being singular

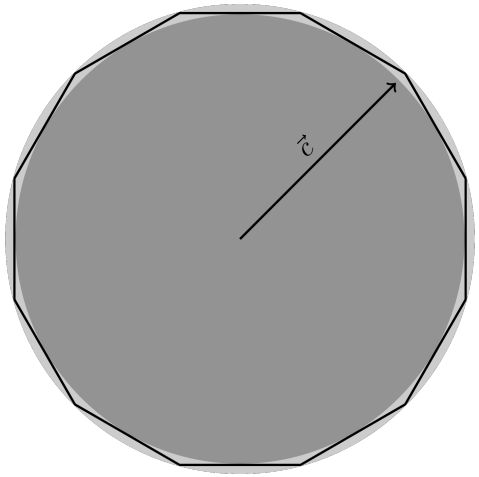
Geometrical interpretation

The condition number is the ratio between the two circles.

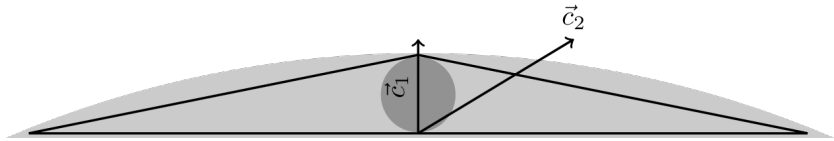
Condition Number
Geometrical interpretation

Which system of equations do you think will have a better condition number?

A:



B:



Example 1
Large matrix coefficient ranges

$$B = \begin{pmatrix} M & 1 \\ 0 & \frac{1}{M} \end{pmatrix} \text{ has inverse } B^{-1} = \begin{pmatrix} \frac{1}{M} & -1 \\ 0 & M \end{pmatrix}$$

$$\Rightarrow \|B\| \geq M \text{ and } \|B^{-1}\| \geq M$$

$$\Rightarrow \kappa(B) = \|B\| \cdot \|B^{-1}\| \geq M^2$$

$\|A\| = \|A\|_\infty \geq \max \sum_i |a_{ij}|$ (maximum of sum of absolute values of elements in rows).

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$$\Rightarrow \|B\| \geq M \text{ and } \|B^{-1}\| \geq M$$

$$\Rightarrow \kappa(B) = \|B\| \cdot \|B^{-1}\| \geq M^2$$

- Avoid unnecessarily large `big M` values
- Avoid unnecessarily small units of measurement
- Avoid combining both of these!

$\|A\| = \|A\|_\infty \geq \max \sum_i |a_{ij}|$ (maximum of sum of absolute values of elements in rows).

Example 1.2
Large matrix coefficient ranges

What would happen if we replace $\frac{1}{M}$ with just M?

$$B = \begin{pmatrix} M & 1 \\ 0 & M \end{pmatrix} \text{ has inverse } B^{-1} = ?$$

Example 1.2
Large matrix coefficient ranges

What would happen if we replace $\frac{1}{M}$ with just M?

$$B = \begin{pmatrix} M & 1 \\ 0 & M \end{pmatrix} \text{ has inverse } B^{-1} = ?$$

The inverse would become:

$$B^{-1} = \begin{pmatrix} \frac{1}{M} & \frac{-1}{M^2} \\ 0 & \frac{1}{M} \end{pmatrix} \Rightarrow \|B^{-1}\| \geq \frac{1}{M} \Rightarrow \kappa(B) \geq 1$$

Example 2
Inconsistent or unnecessarily truncated data

- Values that should be equal have slight differences
- Example:

```
  c306: 0.41666666 7 x80 – 100 x90 = 0
 [...]
c11360: 0.41666666 6 x80 – 100 x90 = 0
```

- Values appear to be calculated inconsistently
- Values are only calculated to 9 digits → bigger perturbation than necessary

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Inconsistent or unnecessarily truncated data

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```

- Values appear to be calculated inconsistently
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Example 3
Hidden mixtures of large and small coefficients

- $\kappa(B) = \|B\| \cdot \|B^{-1}\|$ is large if $\|B\|$ or $\|B^{-1}\|$ is large
- $\|B^{-1}\|$ can also be large for other reasons
- Cascade or staircase:

$$B = \begin{pmatrix} 1 & -2 & & & \\ & 1 & -2 & & \\ & & \ddots & \ddots & \\ & & & 1 & -2 \\ & & & & 1 \end{pmatrix}, \quad B^{-1} = \begin{pmatrix} 1 & 2 & 4 & \dots & 2^{n-1} \\ & 1 & 2 & \dots & 2^{n-2} \\ & & \ddots & \ddots & \vdots \\ & & & 1 & 2 \\ & & & & 1 \end{pmatrix}$$

Example 3
Hidden mixtures of large and small coefficients

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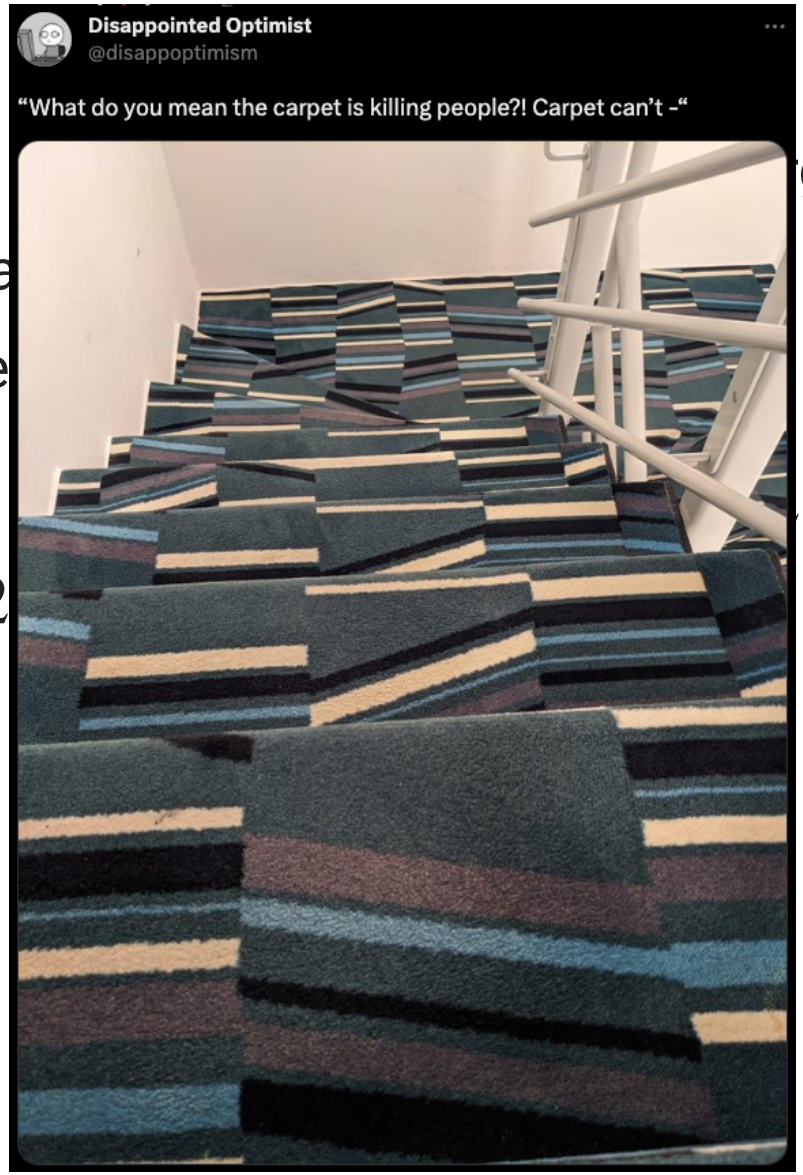
- $\|B^{-1}\|$ can also be la
- Cascade or staircase

$$B = \begin{pmatrix} 1 & -2 & & & & 4 & \dots & 2^{n-1} \\ & 1 & -2 & & & 2 & \dots & 2^{n-2} \\ & & \ddots & \ddots & & \ddots & \ddots & \vdots \\ & & & 1 & -2 & & 1 & 2 \\ & & & & 1 & & & 1 \end{pmatrix}$$



Quick summary

Reasons for a high condition number

- Large spread of coefficients
- Strange properties (e.g. cascade)
- Almost linear dependent rows or columns

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How can we fix this?

Quick summary

Reasons for a high condition number

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How can we fix this?

Important

The condition number is a **property of the problem**! Not the solution process.

Probably reformulation but wait and see!

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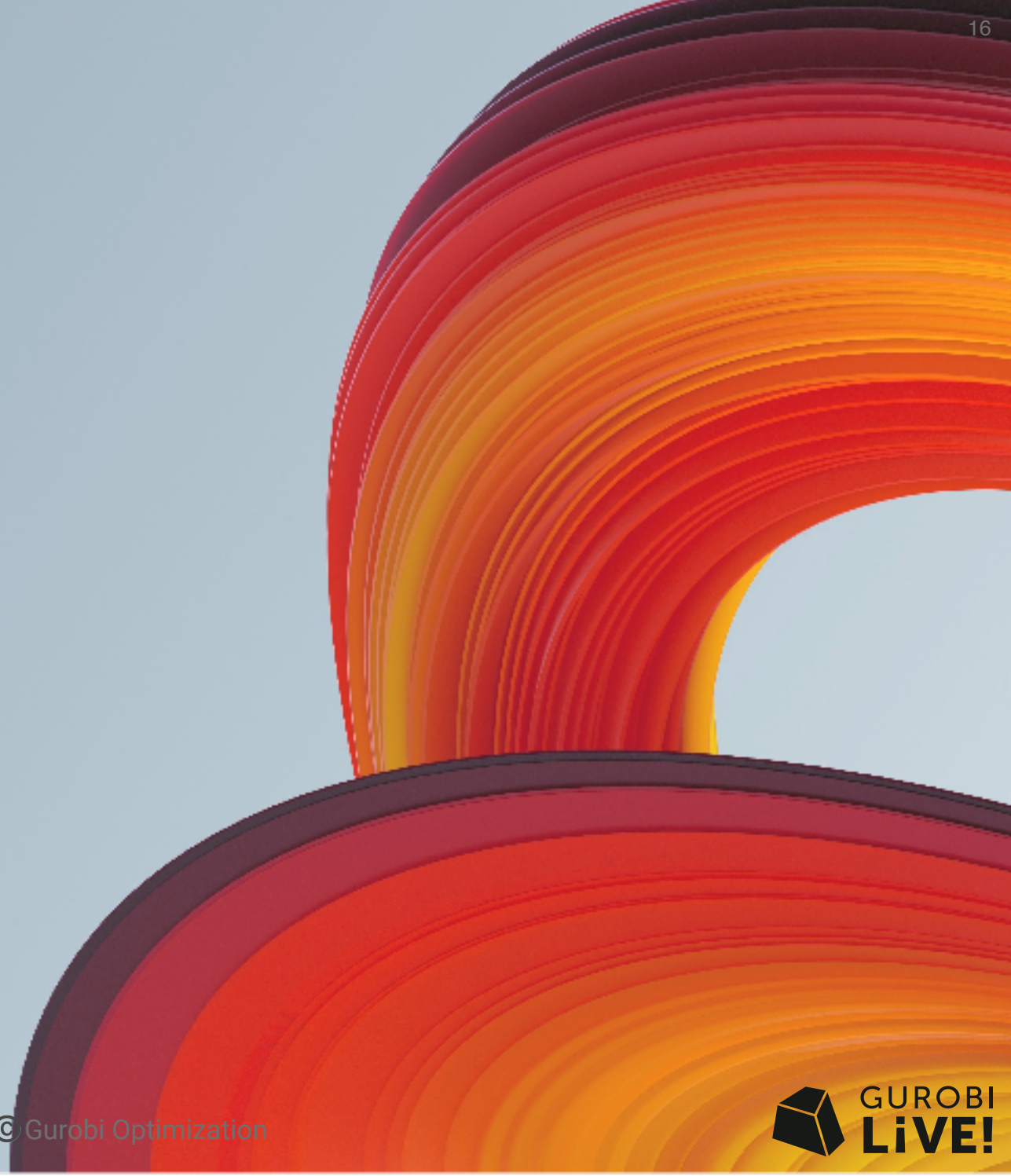


Why is this relevant?
LPs

A solution to an LP requires:

1. A basis
2. Feasibility verification or proof of infeasibility
3. Optimality verification or proof of unboundedness

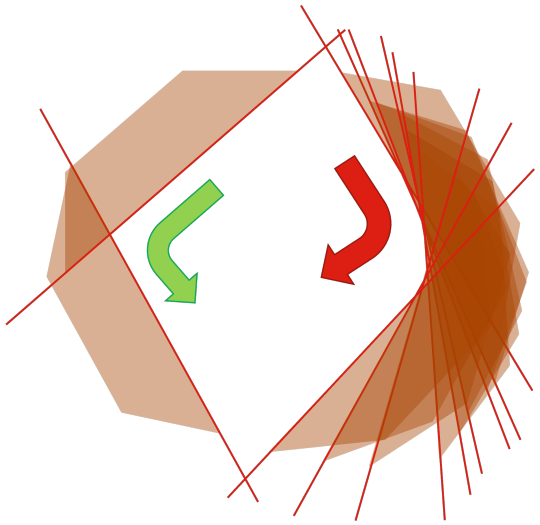
Requires solving two linear systems of equations (with basis matrix B):

- Primal solution: $x_B = B^{-1}(b - A_N x_N)$
- Dual solution: $\pi^T = c_B^T B^{-1}$

Why is this relevant?
LPs

In the simplex algorithm, large condition numbers may lead to:

- Incorrect selection of pivots due to numerical errors.
- Pivots not found due to numerical errors.
- Excessive iterations.



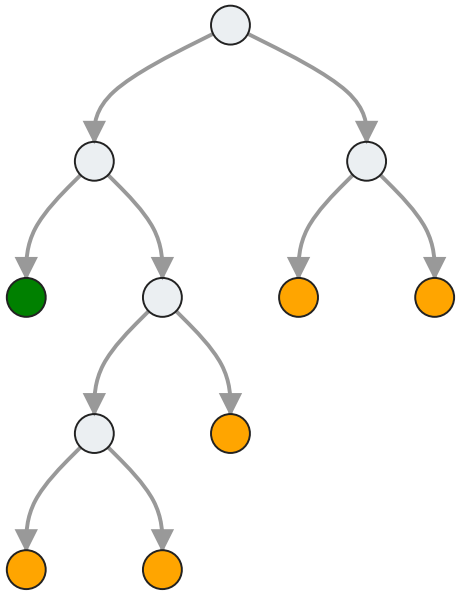
For verification, B is used.

Why is this relevant?
LPs

- A stable implementation tries to avoid running into badly conditioned bases along the way
  - Steepest edge pricing and Harris ratio test (Bixby and Rothberg, 2003).
  - Markowitz tolerance in basis factorization
- May not always be possible

Bixby, R. E., & Rothberg, E. (2003). Solving linear and integer programs. MPI Informatik ADFOCS.

Why is this relevant?
MIPs



- The LP relaxation is solved after every branching decision and cut.
- Pruning, optimality checks (these are hard!).

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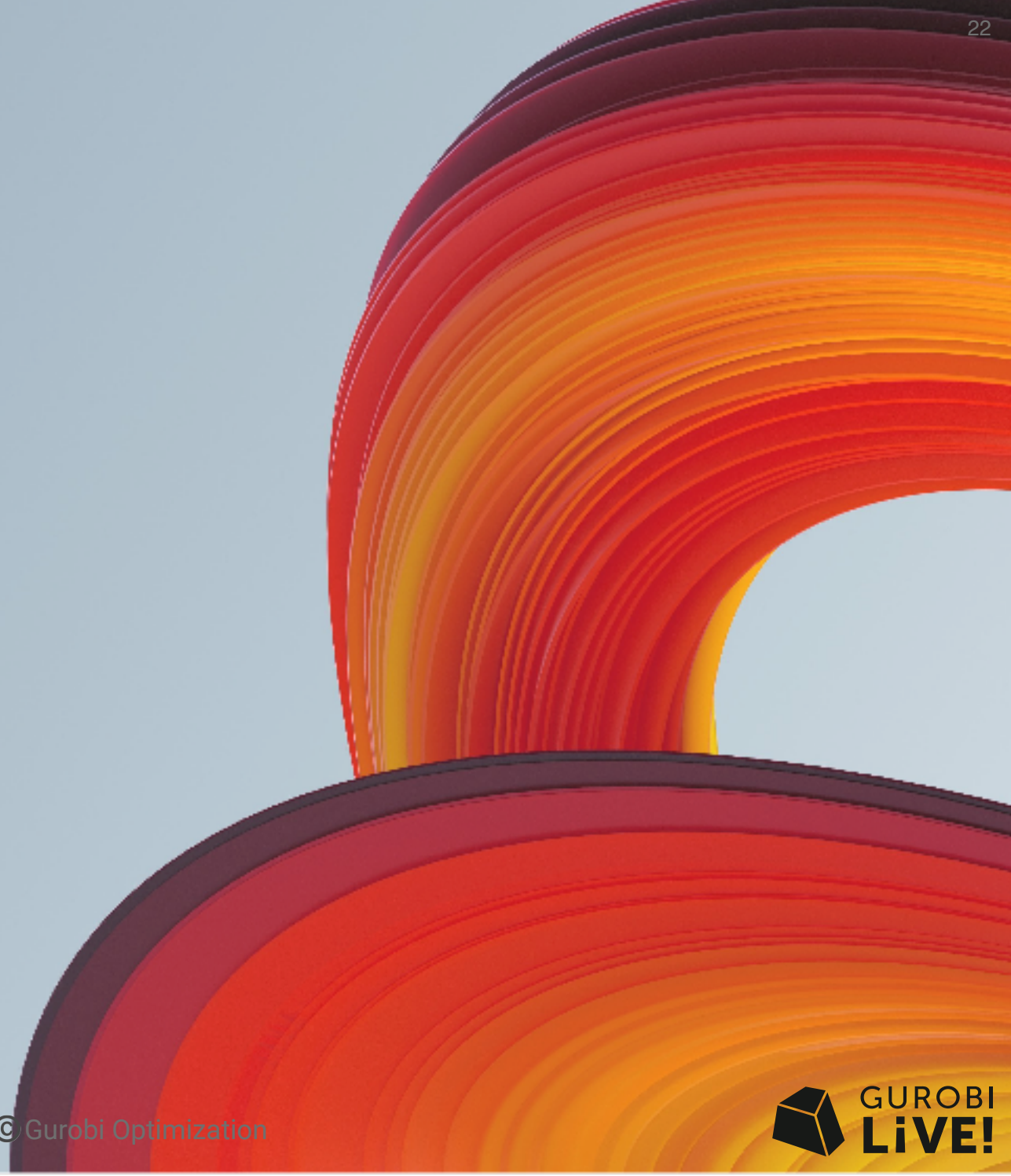

Floating Point Numbers

```
1 + pow(2,-52)
```

```
1.0000000000000002
```

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```
1.0
```

Fails basic mathematical properties:

- Commutativity
- Associativity

```
pow(2,-53) + pow(2,-53) + 1 == 1 + pow(2,-53) + pow(2,-53)
```

```
False
```

Floating Point Numbers

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```
pow(2,-53) + pow(2,-53) + 1 == 1 + pow(2,-53) + pow(2,-53)
```

```
False
```

```
(1 +  pow(2,-53)) + pow(2,-53) == 1 + (pow(2,-53)  + pow(2,-53))
```

```
False
```

Floating Point Numbers

Fails others: Commutativity in multiplication

```
s = [1, 2, 3, 4, 5]
w = [11.11, 22.22, 33.33, 44.44, 55.55]

# Multiply component wise
r1 = 0.0
for i in range(5):
    r1 += s[i] * w[i]

# Multiply component wise (in reverse order)
r2 = 0.0
for i in range(4, -1, -1):
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```

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Who thinks `r1 == r2`?

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```

Who thinks `r1 == r2`?

```
print(f"{r1=}, {r2=}")
print(f"{r1==r2=}")
```

```
r1=611.05,
r2=611.0500000000001
r1==r2=False
```

Representation of Numbers

type	# of bits	mantissa	exponent	arch
`float`	32	23	8	most
`double`	64	52	11	most
`long double`	80	63	15	e.g. x86-64 (intel or AMD)
`long double`	128	112	15	e.g. Power 9

One bit is always reserved for the sign.

Representation of Numbers

`float` representation of 1234.5 (calculator)

Matthias Miltenberger's WorkspaceFloating Point Calculator
Observable

bias is 127 (so exponent is actually 127 + 10 = 137).

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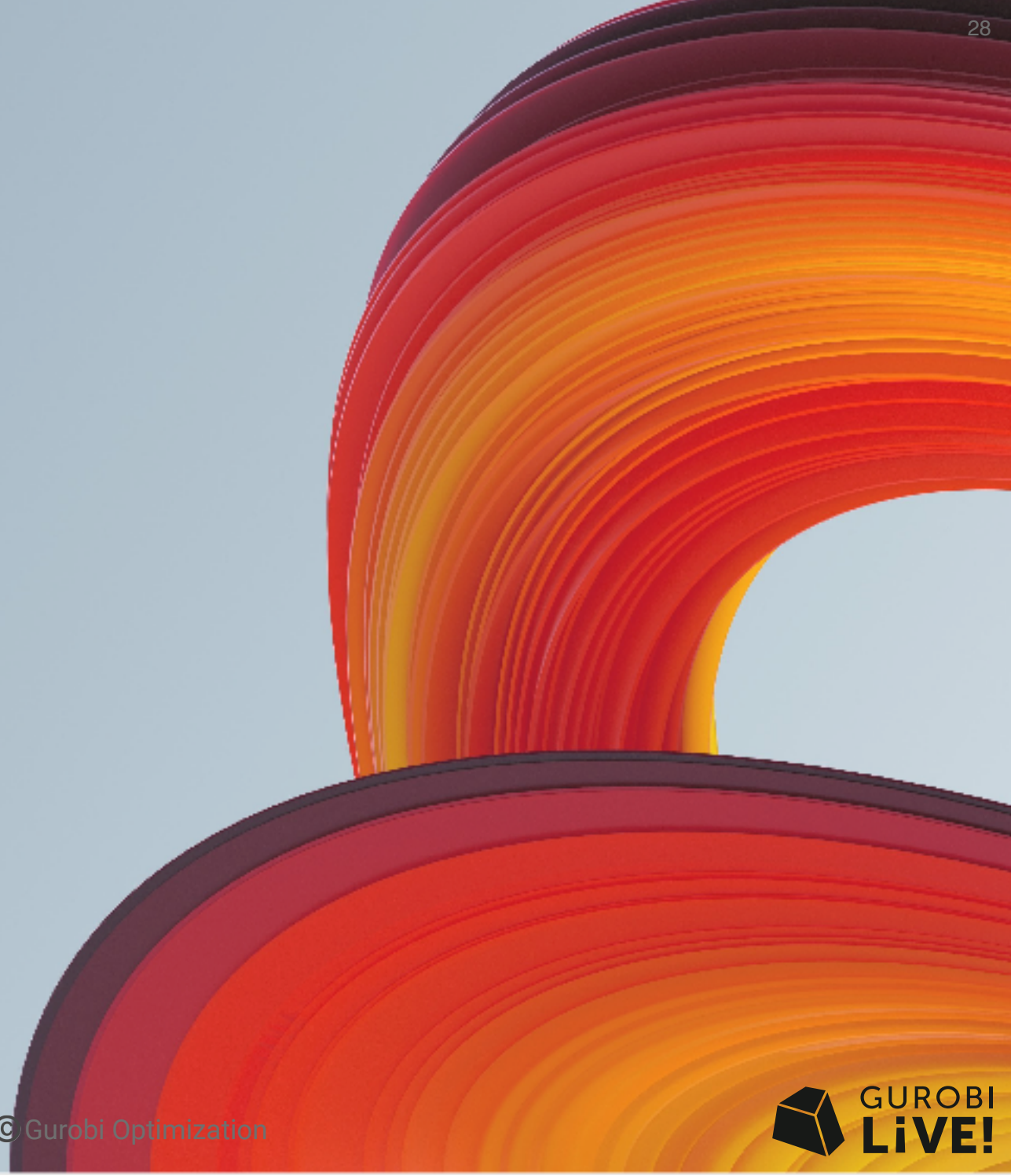

Analyzing

1. Solution and Formulation
2. Optimization
3. Different runs

Analyzing
Solution & Formulation

- Model statistics: `model.printStats()`
- Solution quality: `model.printQuality()`
  - Many more Quality Attributes
- Check the condition number.
  - We will see how later…

Analyzing
Optimization

> ⚠ **Warning**
>
> Look out for warnings!

Prior to optimization

```
Optimize a model with 306204 rows, 579547 columns and 1563587 nonzeros
Model fingerprint: 0x21cdd0c7
Coefficient statistics:
  Matrix range     [1e-05, 2e+08]
  Objective range  [8e-03, 1e+05]
  Bounds range     [1e-03, 3e+09]
  RHS range        [5e-01, 2e+04]
Warning: Model contains large matrix coefficient range
Warning: Model contains large bounds
         Consider reformulating model or setting NumericFocus
         parameter to avoid numerical issues.
```

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Analyzing
Optimization

During LP solving (simplex or barrier)

```
Iteration    Objective       Primal Inf.    Dual Inf.      Time
[...]
   56658   -3.3179831e+03   4.239720e+05   0.000000e+00     20s
Warning: Markowitz tolerance tightened to 0.0625
   67557   -3.2801023e+03   1.974011e+04   0.000000e+00     25s
   69517    3.7944349e+02   6.207667e+04   0.000000e+00     30s
Warning: 1 variables dropped from basis
Warning: switch to quad precision
   78320    1.7027054e+08   1.149199e+04   0.000000e+00     35s
```

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Warning: 1 variables dropped from basis
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```

Check also solution status (can be `suboptimal` or `numerical trouble encountered`)

```
Barrier performed 49 iterations in 0.03 seconds (0.02 work units)
Numerical trouble encountered
```

Analyzing
Optimization

During the MIP node process

```
    Nodes    |    Current Node    |     Objective Bounds      |     Work
Expl Unexpl |  Obj  Depth IntInf | Incumbent    BestBd   Gap | It/Node
[...]
  3252      0 278173.332    0 2120 275179.027 277183.668  0.73%   211  4
  3253      0 278173.020    0 2060 275179.027 277115.904  0.70%   211  4
Relaxation Markowitz Tolerance tightened to 0.5
  3254      0 278173.020    0 2060 275179.027 277115.904  0.70%   211  4
[...]
  3575    295  postponed   43      275179.027 277115.904  0.70%   321  4
  3599    314 infeasible   44      275179.027 277115.904  0.70%   332  4
  3673    391 276640.457   55 1881 275179.027 277115.904  0.70%   328  4
```

Analyzing
Optimization

During the MIP node process

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```

`postponed` nodes may have to be processed later - but hopefully never

Analyzing
Optimization

Post Optimization

```
1  Solved in 131511 iterations and 75.99 seconds (66.11 work units)
2  Optimal objective  1.730551793e+08
3  Warning: unscaled dual violation = 0.0141201 and residual = 0.0111682
```

Analyzing
Optimization

Post Optimization

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1 Solved in 131511 iterations and 75.99 seconds (66.11 work units)
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2 Optimal objective  1.730551793e+08
3 Warning: unscaled dual violation = 0.0141201 and residual = 0.0111682
```

- This can also be checked with the quality attributes or `MaxVio`

Analyzing different runs

Important

Numerical issues can lead to different solutions!

- What happens if we run the same model using different seeds?
  - Does the solution remain the same?
  - Does the time remain similar?
    - This may be due to other factors

Analyzing different runs

Two different solutions for `harp2` (using `MIPGap=0`).

```
2c2
< # Objective value = -73899798
---
> # Objective value = -73899799.3523847
41,42c41,42
< x39 0
< x40 1
---
> x39 1
> x40 0
451,452c451,452
< x449 1
< x450 0
---
> x449 0
> x450 1
697,698c697,698
< x695 1
```

Analyzing different runs

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> x40 0
451,452c451,452
< x449 1
< x450 0
---
> x449 0
> x450 1
697,698c697,698
```

Analyzing different runs

Two different solutions for `harp2` (using `MIPGap=0`).

```
2c2
< # Objective value = -73899798
---
> # Objective value = -73899799.3523847
41,42c41,42
< x39 0
< x40 1
---
> x39 1
> x40 0
451,452c451,452
< x449 1
< x450 0
---
> x449 0
> x450 1
697,698c697,698
```

Model Reformulation

The problem with big-M

```python
import gurobipy as gp

M = 1e10
opts = {"OutputFlag": 0}

with gp.Env(params=opts) as env, gp.Model(env=env) as m:
    c = m.addVar(name="c", lb=0)
    b = m.addVar(name="b", vtype=gp.GRB.BINARY)
    # Fix variables
    c.UB = 1e4
    c.LB = 1e4
    b.UB = 1e-6 # This is technically zero based on the IntFeasTol
    b.LB = 1e-6
    m.addConstr(c <= M*b) # So we really want c = 0 since b = 0
    m.optimize()
    print(f"Is the model optimal? {m.Status==gp.GRB.OPTIMAL}")
```

```
Is the model optimal? True
```

Model Reformulation

The problem with big-M

```python
import gurobipy as gp

M = 1e10
opts = {"OutputFlag": 0}

with gp.Env(params=opts) as env, gp.Model(env=env) as m:
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```

```
Is the model optimal? True
```

⚠ **How can we fix this?**

Model Reformulation

Alternative 1

```python
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M = 1e10
opts = {"OutputFlag": 0}

with gp.Env(params=opts) as env, gp.Model(env=env) as m:
    c = m.addVar(name="c", lb=0)
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    # Fix variables
    c.UB = 1e4
    c.LB = 1e4
    b.UB = 1e-6 # This is technically zero based on the IntFeasTol
    b.LB = 1e-6
    m.addConstr((b == 0) >> (c == 0))
    m.optimize()
    print(f"Is the model optimal? {m.Status==gp.GRB.OPTIMAL}")
```

```
Is the model optimal? False
```

Model Reformulation

Alternative 2

```python
import gurobipy as gp

M = 1e10
opts = {"OutputFlag": 0, "IntFeasTol": 1e-7}

with gp.Env(params=opts) as env, gp.Model(env=env) as m:
    c = m.addVar(name="c", lb=0)
    b = m.addVar(name="b", vtype=gp.GRB.BINARY)
    # Fix variables
    c.UB = 1e4
    c.LB = 1e4
    b.UB = 1e-6 # This is technically zero based on the IntFeasTol
    b.LB = 1e-6
    m.addConstr(c <= M*b) # So we really want c = 0 since b = 0
    m.optimize()
    print(f"Is the model optimal? {m.Status==gp.GRB.OPTIMAL}")
```

```
Is the model optimal? False
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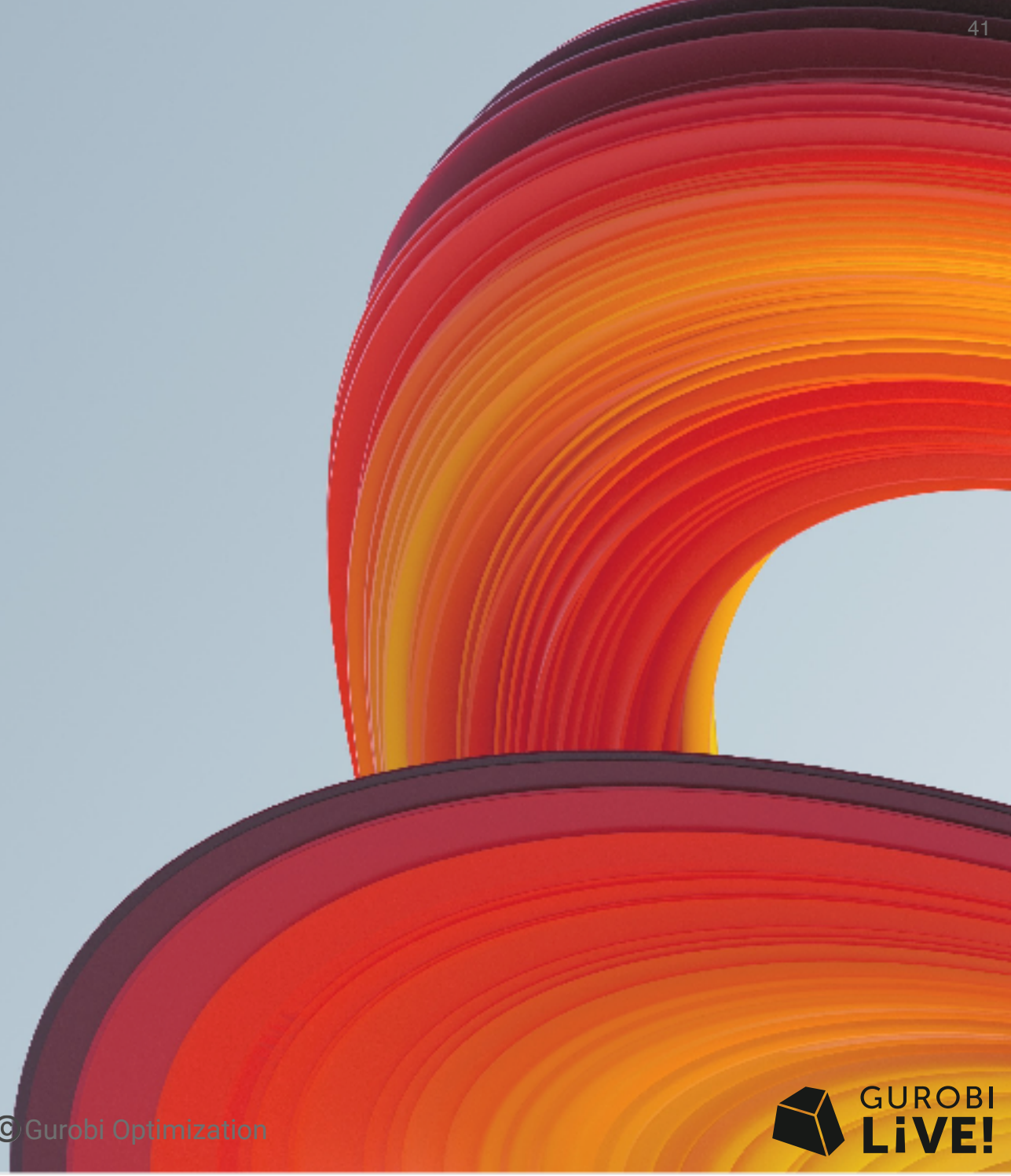
Agenda

1. **Concept**
   - Condition number
   - Why is this relevant?
   - Floating point numbers
2. **Practical recommendations**
   - Analyzing
   - Parameters
   - Summary
3. **Ill-Conditioning Explainer**



There are *many* parameters

… and choosing the right ones is no easy task

Parameter recommendations

`NumericFocus = [0, 1, 2, 3]`

Meta parameter that affects many aspects of the solution process:

Presolve

- Tighter tolerances when fixing variables
- Avoid small coefficients in bound strengthening
- More restrictive checks in aggregator

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Root relaxation

- Simplex
  - Tighten MarkowitzTol tolerance
  - Use Quad precision
- Barrier
  - More conservative step sizes

Parameter recommendations

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Root relaxation

- Simplex
  - Tighten MarkowitzTol tolerance
  - Use Quad precision
- Barrier
  - More conservative step sizes

Cuts

- Only rank-1 MIR Cuts
- Less aggregation
- Disable Gomory Cuts for >= 2
- Disable all Cuts for = 3

Parameter recommendations

`Presolve = [-1, 0, 1, 2]`

⚠ **Presolve typically makes numerically challenging models worse.**

The main culprits are:

- `Aggregate` - may introduce accumulation of numerical errors
  - Try setting this to `0` to deactivate aggregation

Parameter recommendations

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- `Aggregate` - may introduce accumulation of numerical errors
  - Try setting this to `0` to deactivate aggregation
- `Presolve` - changing this to a lower value like `0` or `1` might help
  - Try limiting the number of presolve iterations via `PrePasses`

Parameter recommendations

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⚠ **Presolve typically makes numerically challenging models worse.**

The main culprits are:

- `Aggregate` - may introduce accumulation of numerical errors
  - Try setting this to `0` to deactivate aggregation
- `Presolve` - changing this to a lower value like `0` or `1` might help
  - Try limiting the number of presolve iterations via `PrePasses`
- Scaling: `ScaleFlag` and `ObjScale`
  - Control scaling of the LP and the objective respectively
  - Setting `ScaleFlag = 2` (geometric mean scaling) might help

Parameter recommendations

Root relaxation or pure LP: Simplex

- `MarkowitzTol`: pivoting tolerance (for LU factorization)
  - Larger values increase numerical stability at the expense of sparsity

Remember the warning:

```
Iteration    Objective       Primal Inf.    Dual Inf.      Time
[....]
   56658   -3.3179831e+03   4.239720e+05   0.000000e+00     20s
Warning: Markowitz tolerance tightened to 0.0625
[....]
```

- Gurobi dynamically adjusts this to balance performance and stability

Try setting it to the displayed value right from the start!

Parameter recommendations

Root relaxation or pure LP: Simplex

- `Quad = [-1, 0, 1]`
  - Higher precision at the expense of speed and memory
  - `-1` switches dynamically to quadruple precision if numerical issues arise

```
    69517    3.7944349e+02   6.207667e+04   0.000000e+00     30s
Warning: 1 variables dropped from basis
Warning: switch to quad precision
    78320    1.7027054e+08   1.149199e+04   0.000000e+00     35s
    88724    1.7034414e+08   1.389671e+04   0.000000e+00     40s
```

Parameter recommendations

Root relaxation or pure LP: Barrier

- `BarHomogeneous = [-1, 0, 1]`
  - Default or homogeneous algorithm
  - Homogeneous algorithm is usually a bit slower
  - Homogeneous is better in recognizing infeasibility or unboundedness, or dealing with models that are at the boundary of infeasibility

Parameter recommendations

Root relaxation or pure LP: Barrier

- `BarHomogeneous = [-1, 0, 1]`
  - Default or homogeneous algorithm
  - Homogeneous algorithm is usually a bit slower
  - Homogeneous is better in recognizing infeasibility or unboundedness, or dealing with models that are at the boundary of infeasibility
- `Crossover = [-1, ..., 4]`
  - Helps clean up numerical *dirt* (e.g. small violations)
  - Consider turning it off (`0`) if it stalls, or even setting `Quad` to `1`

Parameter recommendations
MIP

- `Cuts = [-1, ..., 3]`
  - `0`: Disable all cuts (together with `CutPasses = 0` to be completely shut off, may affect heuristics)
  - `1`: Moderate cut generation
  - `2, 3`: Aggressive and even more aggressive

Parameter recommendations
MIP

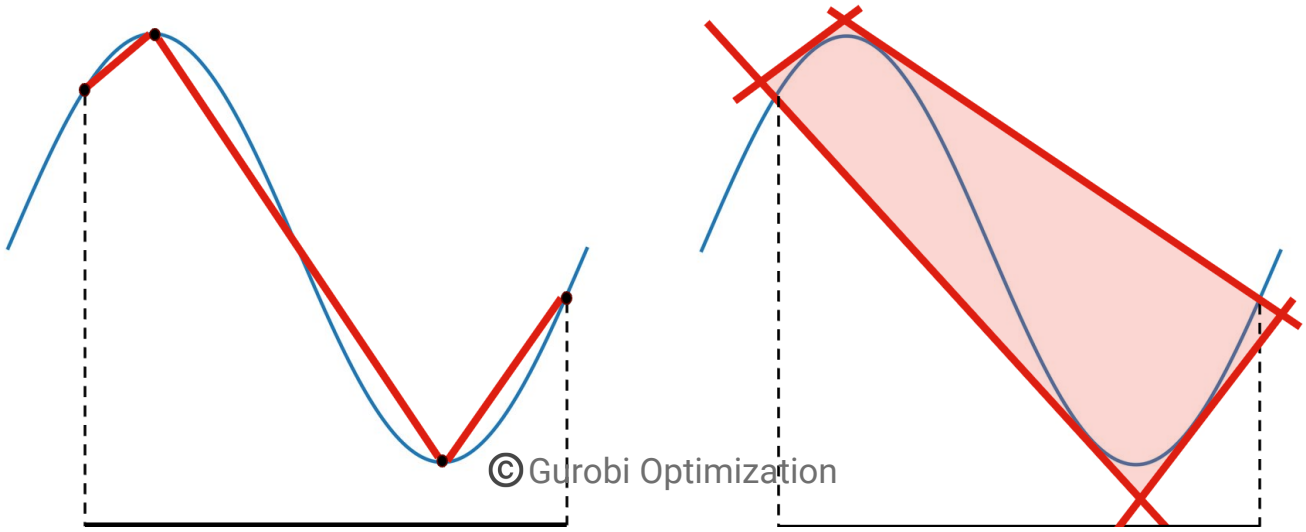
- `Cuts = [-1, ..., 3]`
  - `0`: Disable all cuts (together with `CutPasses = 0` to be completely shut off, may affect heuristics)
  - `1`: Moderate cut generation
  - `2, 3`: Aggressive and even more aggressive
- `GomoryPasses`
  - Most numerically sensitive cutting planes
    - Dense (degrades the LP stability)
    - Uses a row of the basis matrix inverse and requires solving another system of linear equations (risking further issues)

Parameter recommendations
MIP

- `IntegralityFocus = [0, 1]`
  - Enables extra computations to avoid trickle flow (like in the big-M example)
  - Generate a “cleaner” solution with fewer tiny solution values
  - Typically safe to enable but may be expensive

Parameter recommendations
MINLP

- `FuncNonlinear = [0, 1]`
  - Use outer approximation instead of piecewise-linear approximation
  - More accuracy, potentially more expensive
  - When enabled, parameters `FuncPieces`, `FuncPieceError`, `FuncPieceLength`, `FuncPieceRatio` are obsolete
  - only available in Gurobi 11 and when using nonlinear function contraints



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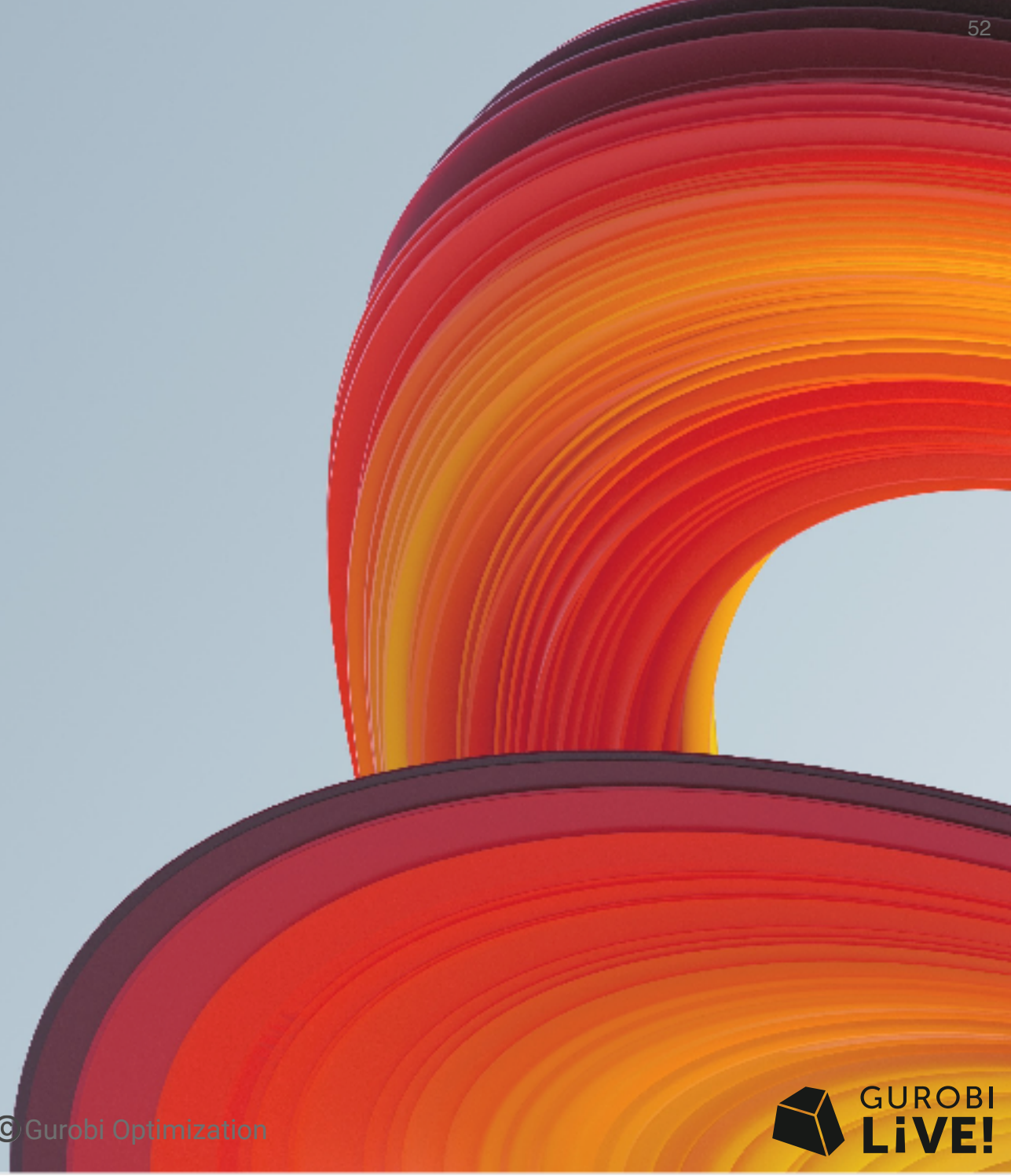

Practical recommendations:
Summary

Analyze:

- Solutions
- Logs
- Different runs

> ⚠ **Warning**
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Practical recommendations:
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Model instance:

- Coefficient statistics
  - Can the ranges be reduced?
- Is presolve helping?

Practical recommendations:
Summary

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- Logs
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Model instance:

- Coefficient statistics
  - Can the ranges be reduced?
- Is presolve helping?

Mathematical formulation:

- Is there an alternative way of formulating the problem?

Practical recommendations:
Summary

> ⚠ **Warning**
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> Turn to parameter tuning only after model improvements have been investigated!

- Try stabilizing the solving behavior via different parameters
- Run multiple seeds and slight variations of the model

Practical recommendations:
Summary

Warning

Turn to parameter tuning only after model improvements have been investigated!

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Tip

Contact the Gurobi Experts team for assistance!

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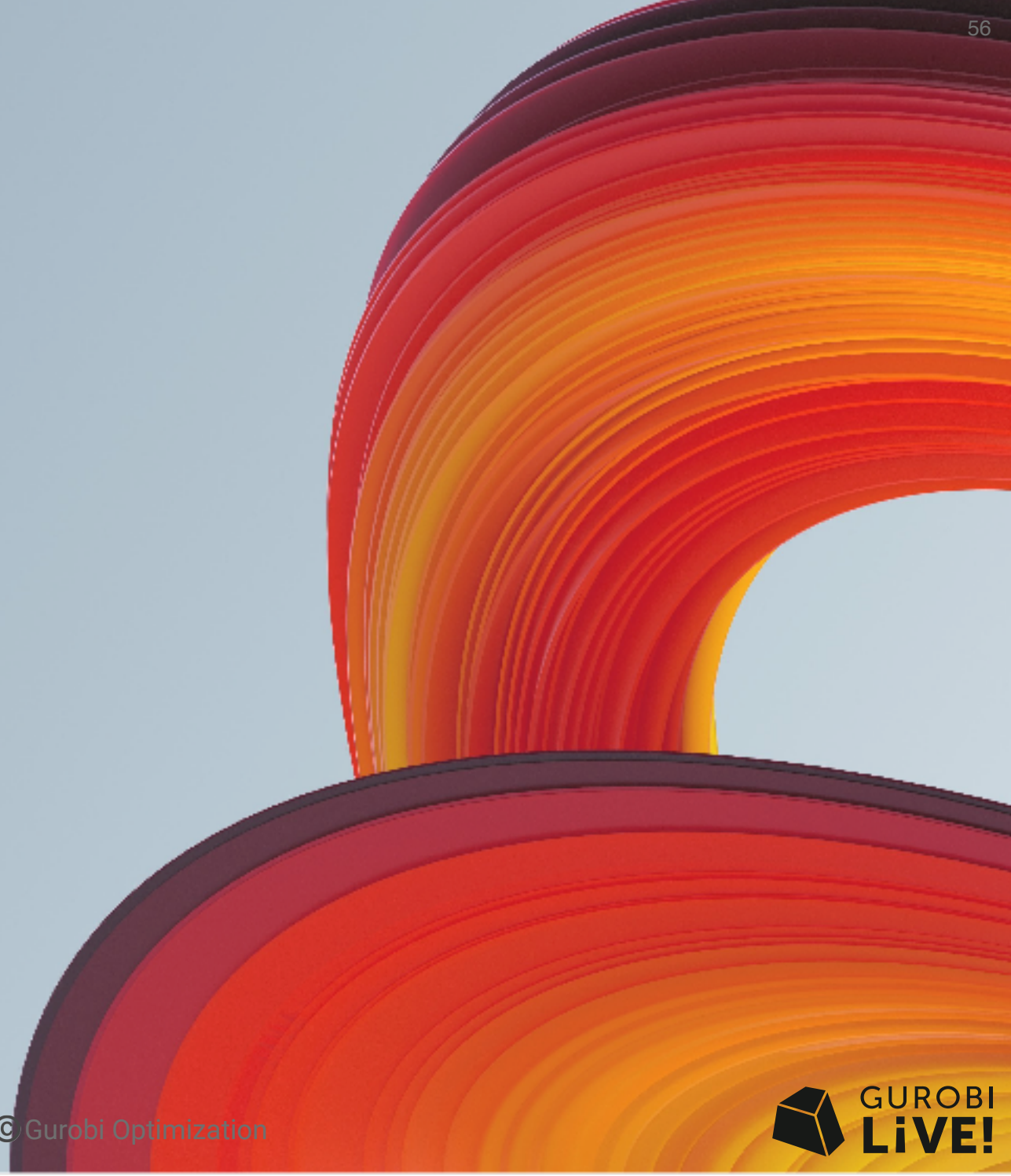

Short intermission: FeasRelax

- Modified model **minimizing the amount** by which the **bounds** and **linear constraints** of the original model are **violated**
- Violation can be measured as:
  - Number of violations (0-norm)
  - Sum of the violations (1-norm)
  - Sum of squares of violations (2-norm)

Infeasible model

$$\begin{aligned} \min\ & c^T x \\ \text{s. t. } & Ax \leq b \\ & x \geq 0 \end{aligned}$$

Feasibility relaxation

$$\begin{aligned} \min & ||(s,u)||_p \\ \text{s. t. } & Ax - s \leq b \\ & x + u \geq 0 \\ & s, u \geq 0 \end{aligned}$$

Tip

Also very helpful to investigate why a model is infeasible!

Basic idea

Approximiate the distance to singularity

Apply `FeasRelax` to minimize the sum of infeasibilities of:

$$
\begin{aligned}
B^T y &= 0 \\
e^T y &= 1 \\
y &\text{ free}
\end{aligned}
$$

- Rows of B with very small y component are filtered out
- Support of y provides a **row-based certificate** of the ill-conditioning
- Add constraint $e^T y = 1$ to prevent $y = 0$

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Third case: Examine angles of pairs of matrix rows and columns

- Vectors u and v are almost parallel if $u^T v - \|u\| \cdot \|v\| < \epsilon$.

Challenges

- The FeasRelax problem is an LP whose structural columns comprise an ill-conditioned basis matrix
  - Set `NumericFocus=3` and `ScaleFlag=2` if row and column ratios are large
  - Provide more custom parameters via `PRM` file

- Too many rows or columns in the output; hard to determine source of ill-conditioning
  - Try different explanation method (row, column, angle)
  - Sometimes one method provides a much smaller explanation

Quick Start Guide

- Installation: `python -m pip install gurobi-modelanalyzer`
- Basic usage:

```
import gurobipy as gp
import gurobi_modelanalyzer as gma
m = gp.read("badmodel.mps")
m.optimize()
print(m.kappa)                          # print kappa of final basis
gma.kappa_explain(m)                    # row-based explanation
gma.kappa_explain(m, expltype="COLS")   # column-based explanation
print(gma.angle_explain(m))             # angle explanation
```

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print(gma.angle_explain(m))             # angle explanation
```

- Modified `afiro` instance for demonstration:

```
Subject To
 R09: - X01 + X02 + X03 = 0
 R09x: - 0.9999999 X01 + X02 + X03 = 0
[...]
```

Output on maliciously modified `afiro`

- `kappa_explain()` will generate a new LP or MPS file, containing the ill-conditioning certificate:
- `afiro_kappaexplain.lp`:

```
Minimize
  0 X36 + 0 X04 + 0 X15 + 0 X16 + 0 X26 + 0 X38 + 0 X37
Subject To
 GRB_Combined_Row: 0.030386883604417606 X23 + 4.80518e-10 X01
   - 4.65661e-10 X03 = 0
 (mult=2696322.968477607)R09x: - 0.999999900000001 X01 + X03 = 0
 (mult=-2696322.6896988587)R09: - X01 + X03 = 0
 (mult=0.2787787486643817)X46: - X03 + 0.109 X22 <= 0
 (mult=0.030386883604417606)R19: X23 - X22 + X24 + X25 = 0
 (mult=0.030386883604417606)X45: - X25 <= 0
 (mult=0.030386883604417606)X48: 0.301 X01 - X24 <= 0
Bounds
End
```

Output on maliciously modified `afiro`

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- `afiro_kappaexplain.lp`:

```
Minimize
  0 X36 + 0 X04 + 0 X15 + 0 X16 + 0 X26 + 0 X38 + 0 X37
Subject To
 GRB_Combined_Row: 0.0303868836044176 X23 + 4.80518e-10 X01
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Bounds
End
```

Angle explanation on modified `afiro` instance

- `angle_explain()` returns a list of tuples:
  1. almost parallel rows
  2. almost parallel columns
  3. associated model

```
(
  [(<gurobi.Constr R09>, <gurobi.Constr R09x>)],
  [],
  <gurobi.Model Continuous instance basismodel:
    28 constrs, 28 vars, No parameter changes>
)
```

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"Real" example: neos-1603965 (MIPLIB 2017)

- Use `model.relax()` to relax integrality conditions
- Final condition number `1.60000000016e+21` (original model)
- `angle_explain()` does not reveal any almost parallel rows or columns

```
Subject To
 GRB_Combined_Row: = 1.000000015655
 (mult=0.999999999815)R24991: C8008 - C8009 >= 0
 (mult=9.9999999981499e-11)R9004: - 1e+10 C8008 <= 0
 (mult=9.9999999981499e-11)R13004: - 0.15 C7002 + 1e+10 C8009 <= 1e+10
 (mult=1.4999999997225e-11)R1030: C3030 - C7001 <= 0
 (mult=-1.4999999997225e-11)R1966: C3966 - C7001 <= 0
 (mult=-1.4999999997225e-11)R2966: C4966 - C7002 <= 0
 (mult=-1.4999999997225e-11)R28014: C0030 + C3030 = 7165
 (mult=1.4999999997225e-11)R28950: C0966 + C3966 + C4966 = 8221
 (mult=1.12499999982e-11)R0030: 1.3333333333 C0030 = 0
 (mult=-1.12499999982e-11)R0966: 1.3333333333 C0966 = 0
```

- Don't ignore rows with small y: `big-M` values of 10^{10} are the issue here

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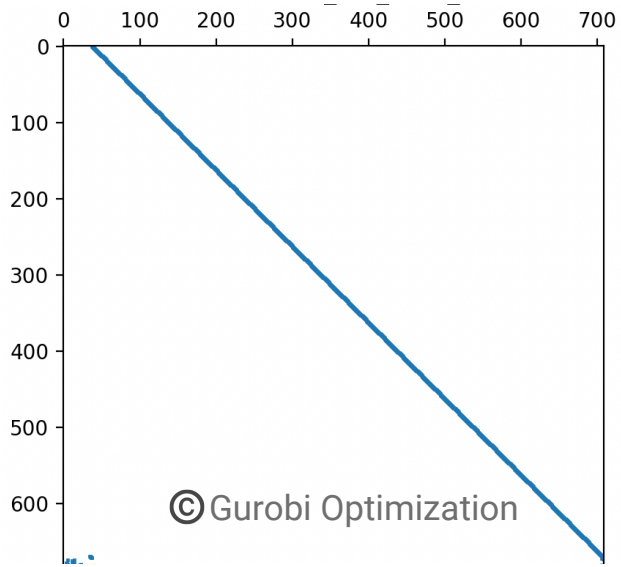
- Don't ignore rows with small y: `big-M` values of 10^{10} are the issue here

More functionality

- `converttofractions()`:
  - takes a list of decimal values and tries to represent them as fractions
  - useful to clean-up “dirty” coefficients
- `matrix_bitmap()`:
  - wrapper for `spy()` (`matplotlib`) to inspect sparsity pattern
  - useful to identify cascades



Summary

- Track down reasons for numerical issues
- Open-source (Apache-2.0) on GitHub:
  https://github.com/Gurobi/gurobi-modelanalyzer
- Install via pip:
  ```
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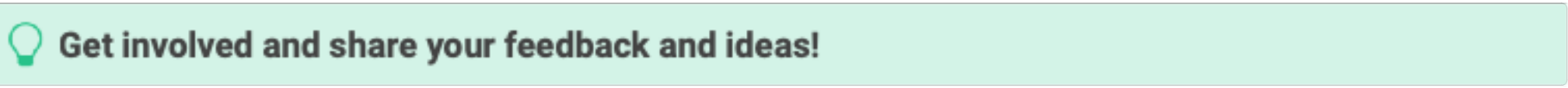
- Currently only supports LPs (use `model.relax()` to inspect root LP of MIPs)



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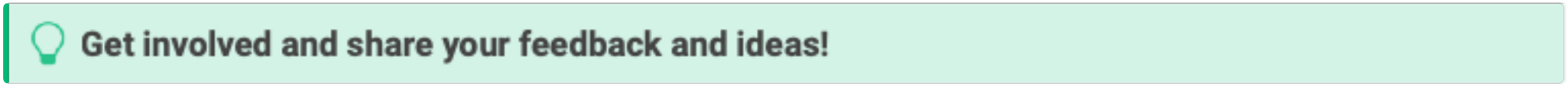

Thank You!

www.gurobi.com

Questions? Comments? Remarks?

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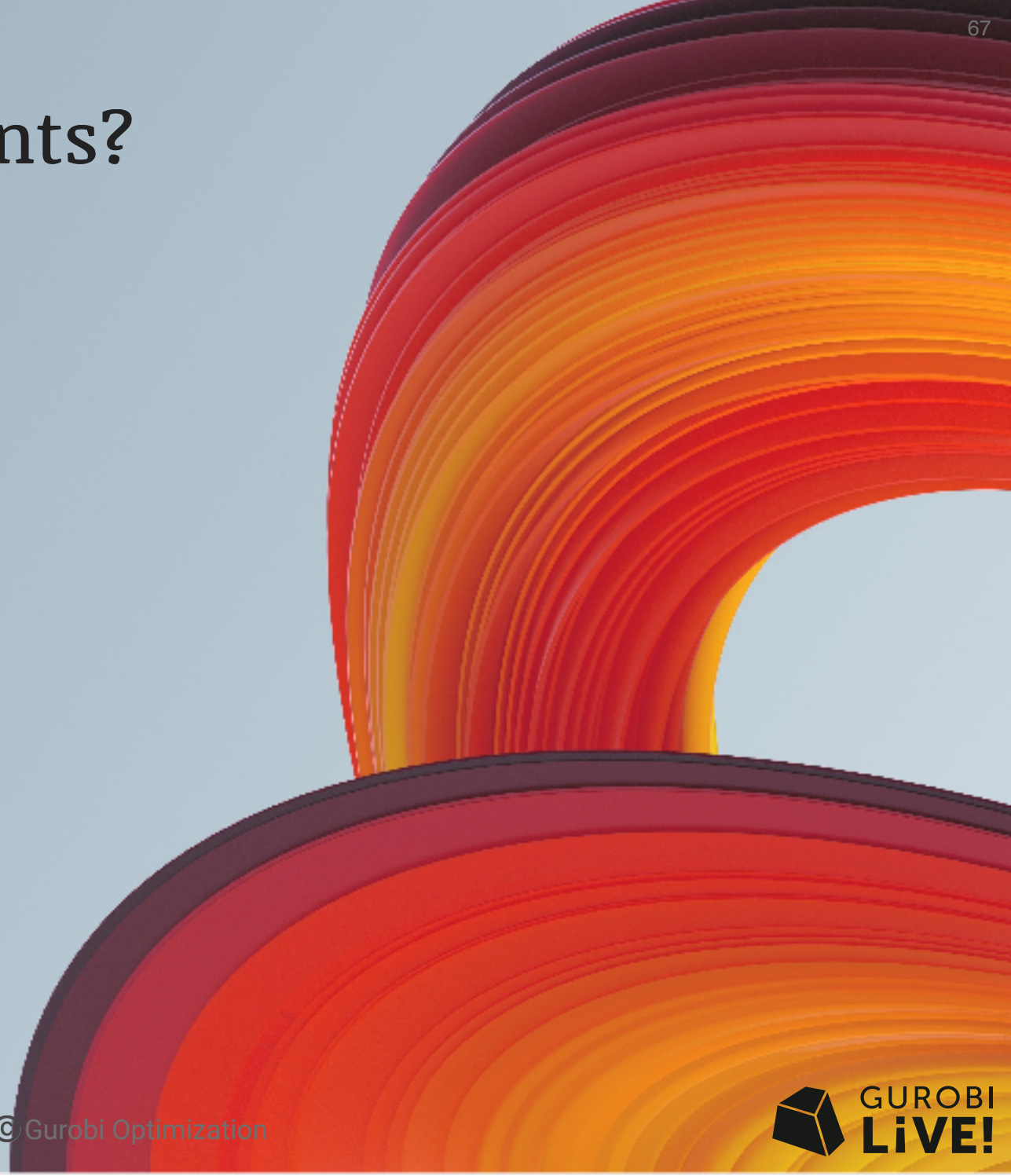

Further Material

- Klotz, E. (2014). *Identification, assessment, and correction of ill-conditioning and numerical instability in linear and integer programs*. In Bridging Data and Decisions (pp. 54-108). INFORMS.
- Gurobi Tech Talk: *Converting Weak to Strong MIP Formulations*, YouTube, Materials.
- Gurobi Holiday Tech Talk: *Prevent the Grinch from Stealing Your Optimization Projects*, YouTube, Materials.