May 4 – 11AM ET | ACADEMIC WEBINAR

The AC Optimal Power Flow Problem

Reviewing the AC Optimal Power Flow Problem and Affiliated Open-Source Packages and Algorithmic Enhancements

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Columbia University

GUROBI OPTIMIZATION

Outline For This Talk

- Motivation, and: what is electrical power?
- What is ACOPF?
- How is ACOPF used in practice?
- Optimization details!!!
- Gurobi and ACOPF.
- Questions welcome. During and after the talk.

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My View: Practical Optimization at a Crossroads

- Current and past areas of interest: logistics, transportation, supply chain.
- These areas will remain relevant, but …
- The future: heavy engineering and hard science.
- Very nonlinear, complex models that embody hard, inflexible rules.
- Very large scale, high level of modeling detail, myriad details in complex systems.
- Demanding performance requirements: must get good solutions fast.
- Are our algorithms up to the task?

POWER GRIDS

Power grids are changing

- Renewables
- Grids are being updated regardless of renewables
- Smart loads
- Demand response
- Local generation

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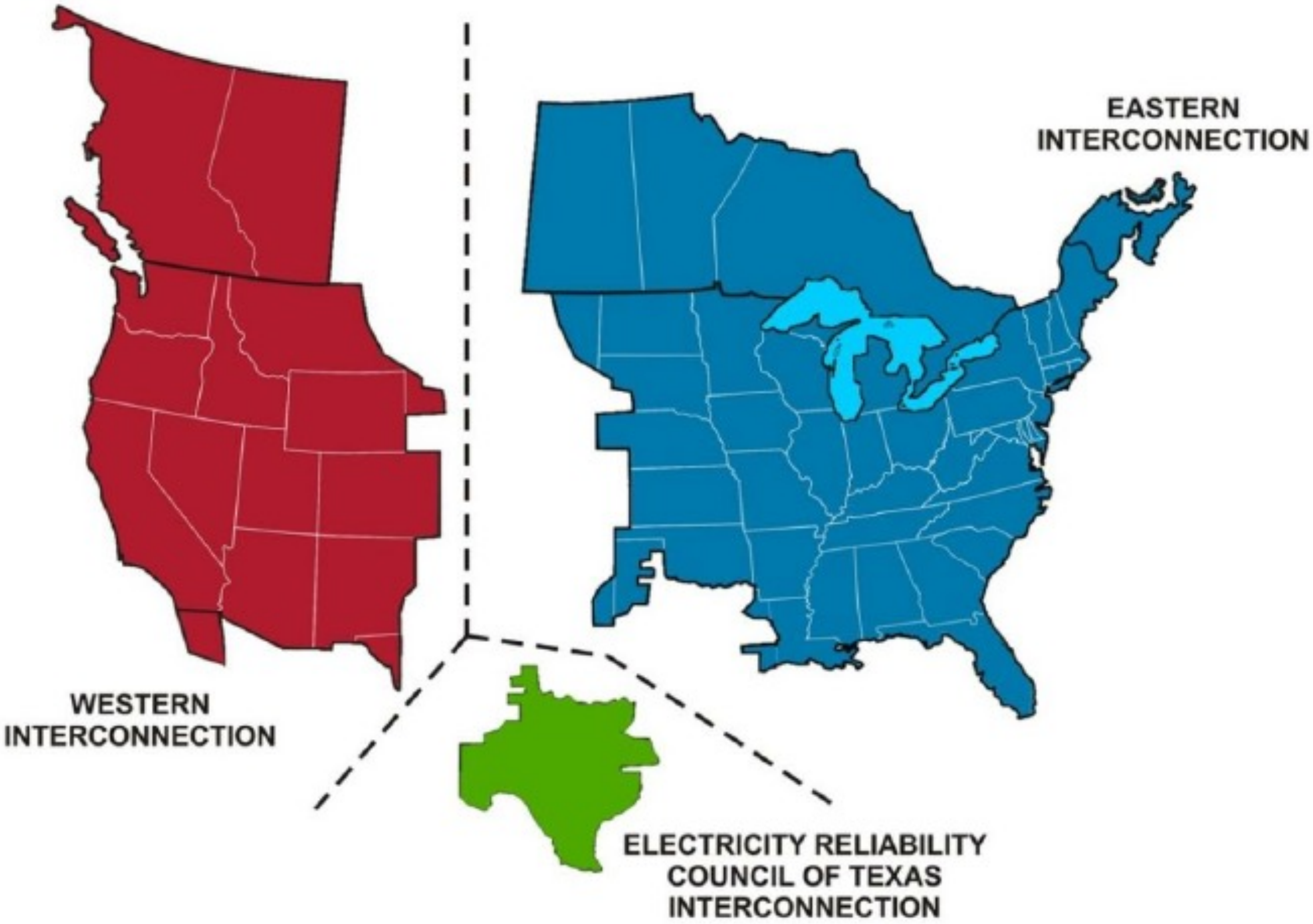
North American Transmission Map

- Three separate electrical circuits
- Except for Texas, too large to control as one unit



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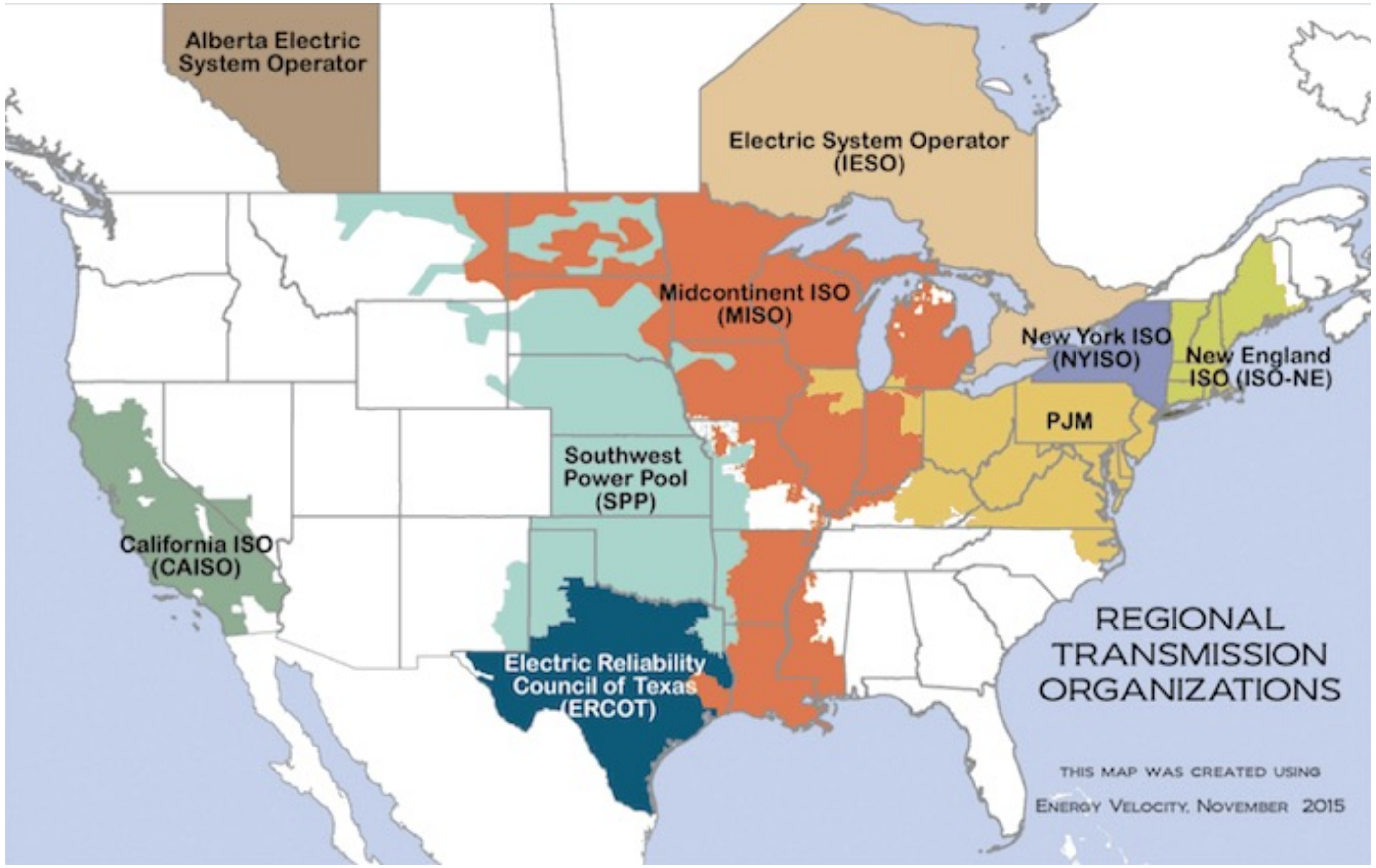
ISOs: Independent System Operators
RTOs: Regional Transmission Operators

- Each ISO/RTO controls the grid in its territory
- Power can travel between neighboring entities, but in a constrained manner
- Each entity operates the markets (energy, capacity, and reserves) in its territory



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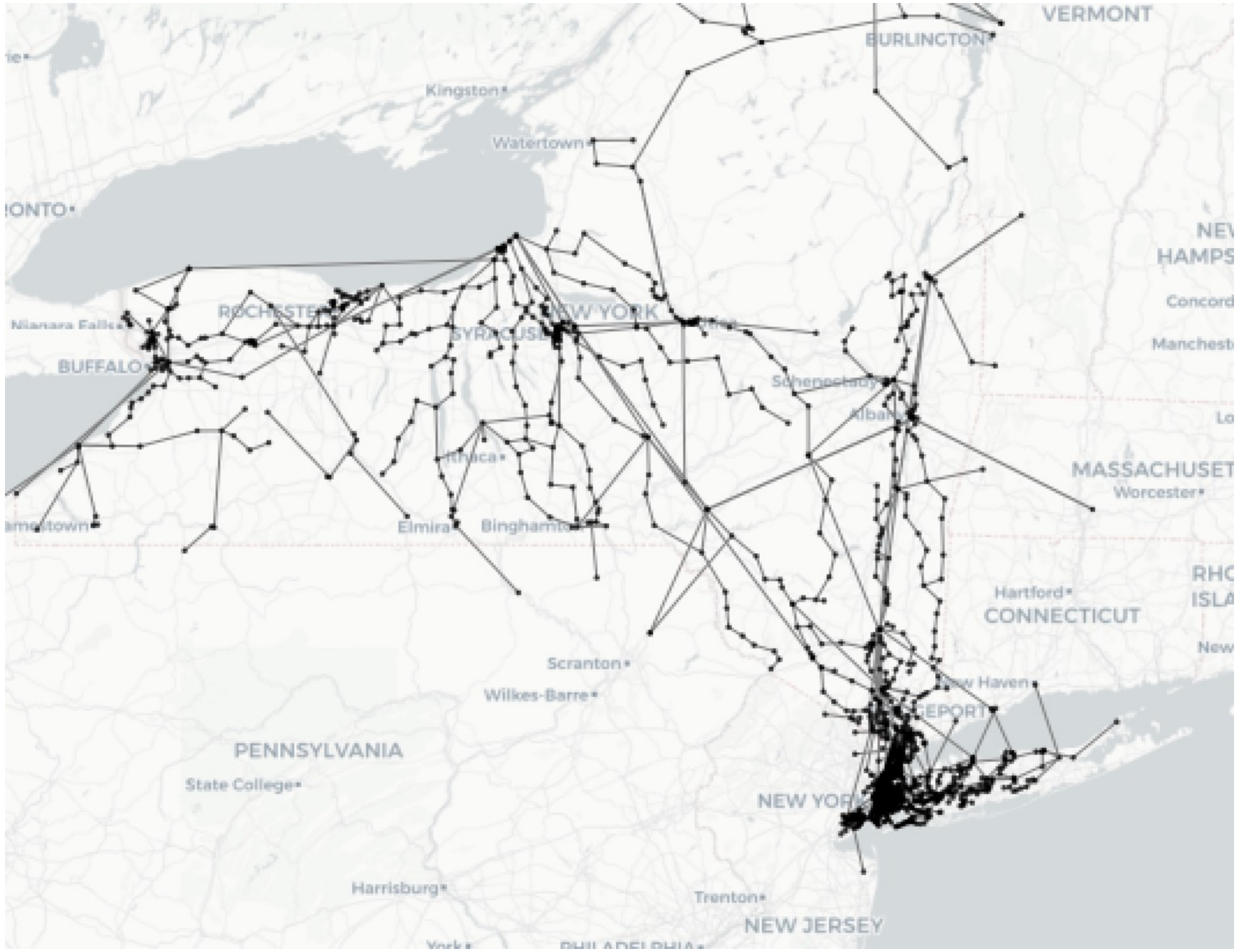

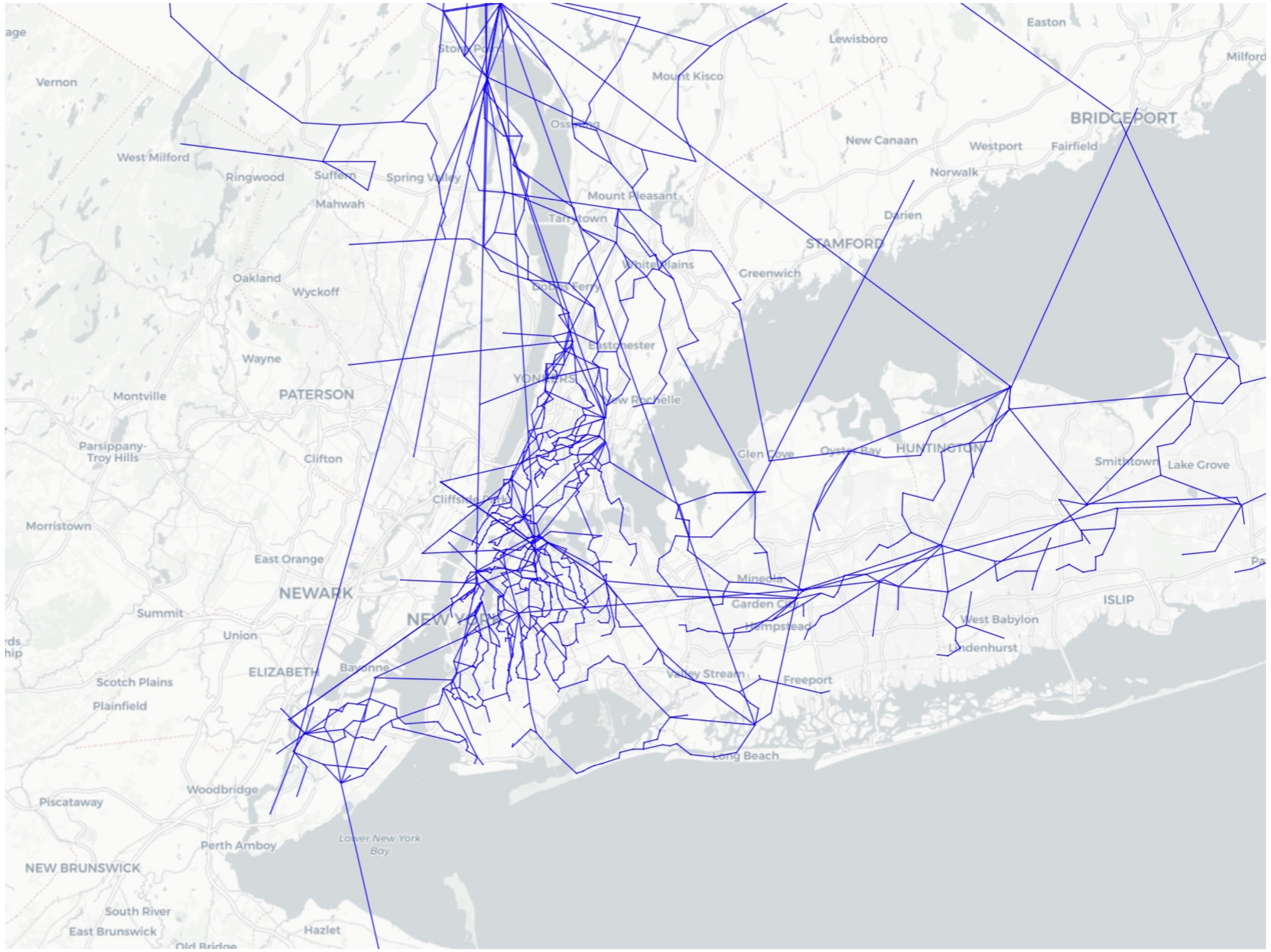

NY system:

- 1814 buses
- 500+ generators

This is not a large system

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Terminology

- What is energy?
- What is power?
- What is electrical current?
- What is voltage?
- Alternating current, voltage, power?
- Basic science goes back hundreds of years: Gauss, Coulomb, Faraday, Ampere, Lorentz, Maxwell
- Modern engineering details are no less complex!

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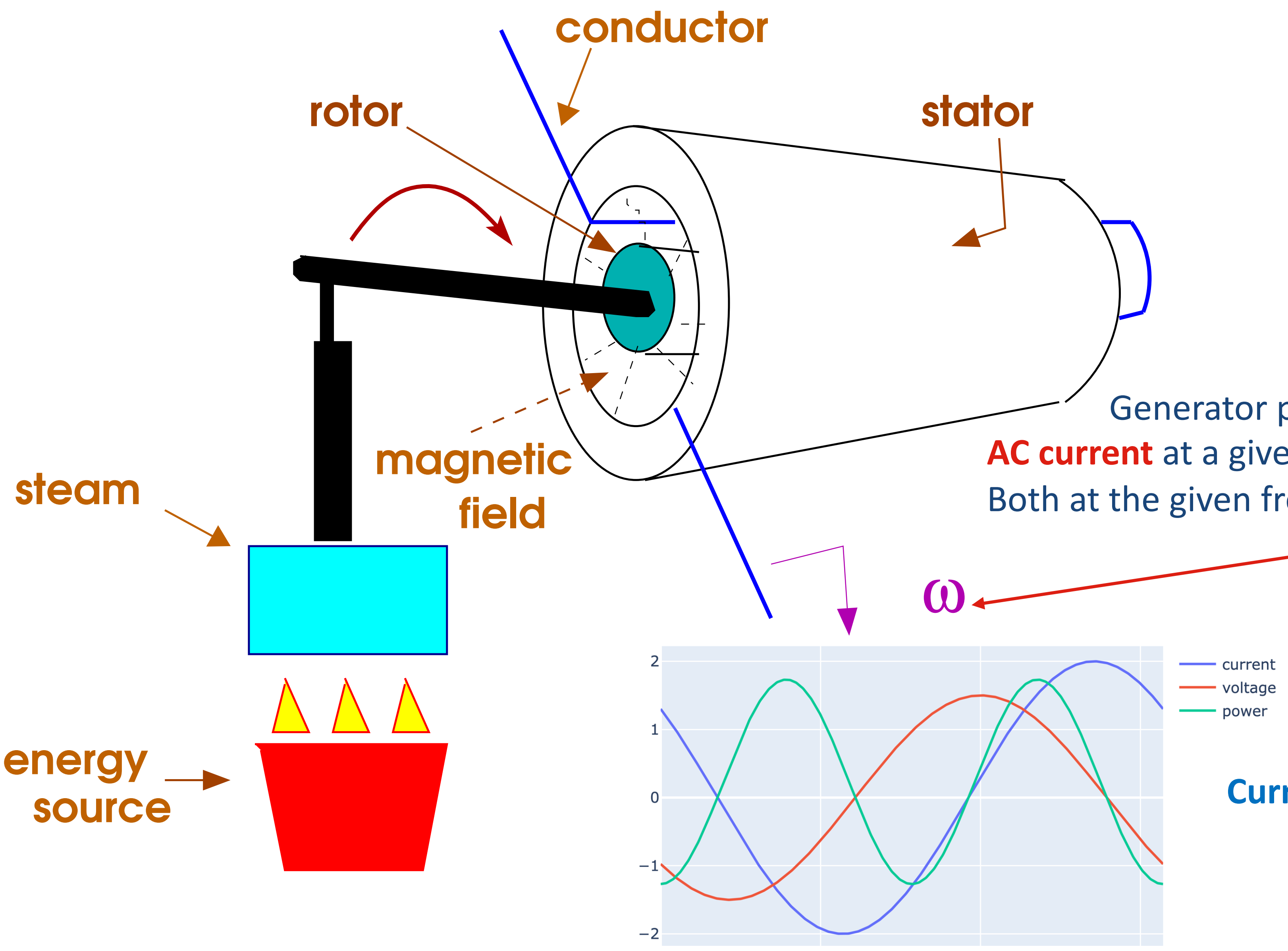

Generator produces
AC current at a given **AC voltage**
Both at the given frequency (60 Hz or 50 Hz)

Current x **Voltage** = **Power**

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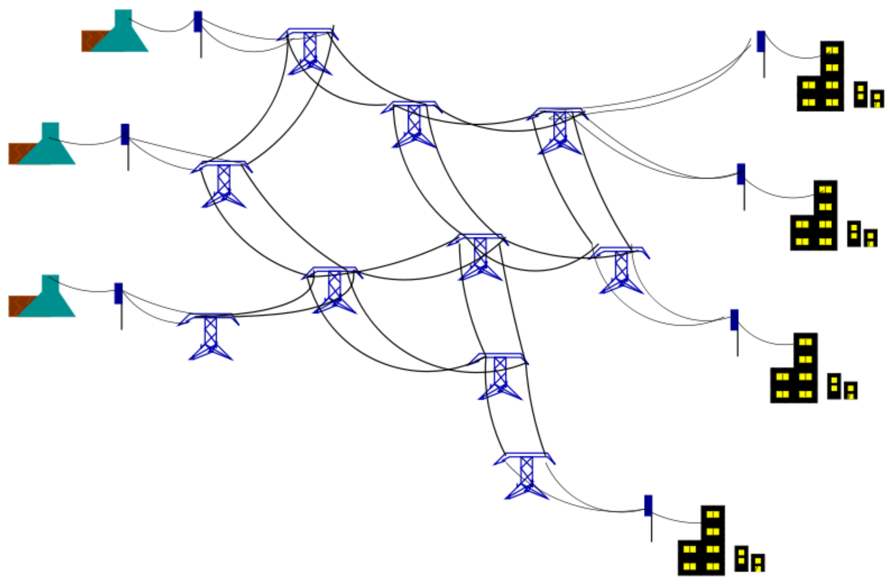
Standard ACOPF

We are given a power system, i.e., a network of

- Generators
- Power lines and transformers
- Buses (nodes)
- Each bus has a load, i.e., numerical demand for power generators, lines, transformers and buses (nodes) with power demands



Objective: meet demands at minimum cost

Cost incurred at generators

Note: power flows following **laws of physics**

A formal textbook statement of standard ACOPF

Minimize cost of generation: $\sum_{g \in \mathcal{G}} F_g(P^g)$

- Here, $\mathcal{G}$ is the set of generators
- P^g is the (active) power generated at g
- F_g is generation cost at g – convex, piecewise-linear or quadratic
  Example: $F_g(P) = 3P^2 + 2P$

Constraints:

- PF (power flow) constraints: choose voltages so that network delivers power from generators to the loads, following **AC power flow laws**
- Voltage magnitudes are constrained
- Power flow on any line km cannot be too large
- The output of any generator is limited

$V_k = |V_k| e^{j\theta_k}$

complex power **injected** into **km** at **k**

$V_m = |V_m| e^{j\theta_m}$

$\theta_{km} \doteq \theta_k - \theta_m$

$$\boldsymbol{S_{km}} = (G_{kk} - jB_{kk})\,|\boldsymbol{V_k}|^2 + (G_{km} - jB_{km})\,|\boldsymbol{V_k}||\boldsymbol{V_m}|(\cos \boldsymbol{\theta_{km}} + \boldsymbol{j} \sin \boldsymbol{\theta_{km}})$$

complex power **injected** into **km** at **k**

OR,

$$Y = \begin{pmatrix} G_{kk} + jB_{kk} & G_{km} + jB_{km} \\ G_{mk} + jB_{mk} & G_{mm} + jB_{mm} \end{pmatrix}$$

Admittance matrix for line km

$$I_{km} = Y \begin{pmatrix} V_k \\ V_m \end{pmatrix}$$

Ohm's Law

Complex current

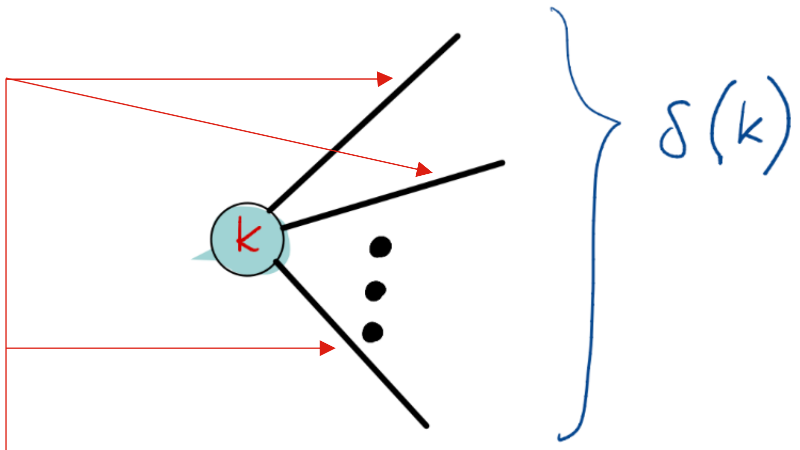
$$S_{km} = V_k I_{km}^*$$

$$S_{km} = V_k \left[Y \begin{pmatrix} V_k \\ V_m \end{pmatrix} \right]^*$$

$$V_k = |V_k| e^{j\theta_k} \qquad \theta_{km} \doteq \theta_k - \theta_m \qquad V_m = |V_m| e^{j\theta_m}$$

$$S_{km} = (G_{kk} - jB_{kk})\,|V_k|^2 + (G_{km} - jB_{km})\,|V_k||V_m|(\cos\theta_{km} + j\sin\theta_{km})$$

S_{km} != $-S_{mk}$



"Reactive" power generation/demand at **k**

$$\sum_{km \in \delta(k)} \boldsymbol{S_{km}} = \left(\sum_{i \in \mathcal{G}(k)} \boldsymbol{P_i^g} - P_k^d\right) + j\left(\sum_{i \in \mathcal{G}(k)} \boldsymbol{Q_i^g} - Q_k^d\right)$$

LHS = complex power injected into grid at **k**

Real power demand at **k**

Total real ("active") power generated at **k**

Basic ACOPF

Minimize $\sum_{i \in \mathcal{G}} F_i(\, \boldsymbol{P_i^g})$

with constraints:

$\forall$ branch km: $\quad \boldsymbol{S_{km}} =$
$(G_{kk} - jB_{kk})\ |\boldsymbol{V_k}|^2 + (G_{km} - jB_{km})\ |\boldsymbol{V_k}||\boldsymbol{V_m}|(\cos \boldsymbol{\theta_{km}} + \boldsymbol{j} \sin \boldsymbol{\theta_{km}})$

$$\sum_{km \in \delta(k)} \boldsymbol{S_{km}} = (\sum_{i \in \mathcal{G}(k)} \boldsymbol{P_i^g} - P_k^d) + j(\sum_{i \in \mathcal{G}(k)} \boldsymbol{Q_i^g} - Q_k^d)$$

Power flow limit on branch km:
$|\boldsymbol{S_{km}}|^2 = \mathcal{R}e(\, \boldsymbol{S_{km}})^2 + Im(\, \boldsymbol{S_{km}})^2 \leq U_{km}$

Voltage limit on bus k:
$V_k^{\min} \leq |\boldsymbol{V_k}| \leq V_k^{\max}$

Generator output limits:
$P_i^{\min} \leq \boldsymbol{P_i^g} \leq P_i^{\max}$

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But there is an equivalent formulation as a **QCQP** (Quadratically Constrained Quadratic Program)

$V_k = |V_k| e^{j\theta_k}$

$V_m = |V_m| e^{j\theta_m}$

$\theta_{km} \doteq \theta_k - \theta_m$

$$S_{km} = (G_{kk} - jB_{kk})\,|V_k|^2 + (G_{km} - jB_{km})\,|V_k||V_m|(\cos\theta_{km} + j\sin\theta_{km})$$

$$S_{km} = V_k \left[Y \begin{pmatrix} V_k \\ V_m \end{pmatrix} \right]^*$$

$$Y = \begin{pmatrix} G_{kk} + jB_{kk} & G_{km} + jB_{km} \\ G_{mk} + jB_{mk} & G_{mm} + jB_{mm} \end{pmatrix}$$

Admittance matrix for line km

$$V_i = e_i + j f_i$$

Use rectangular coordinates for voltages

Exploring the Power Flow Solution Space Boundary

Ian A. Hiskens, *Senior Member* and Robert J. Davy

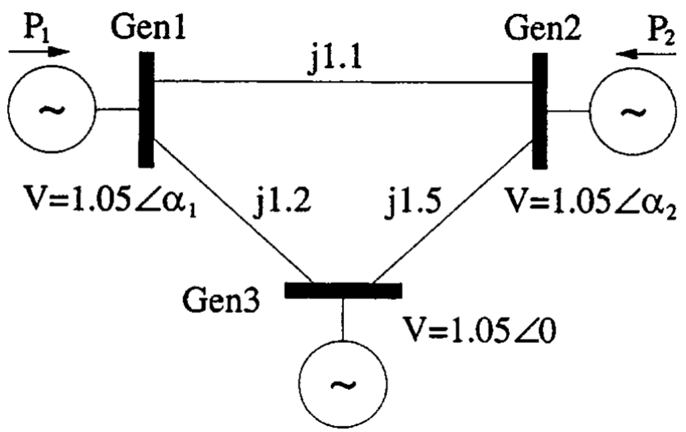
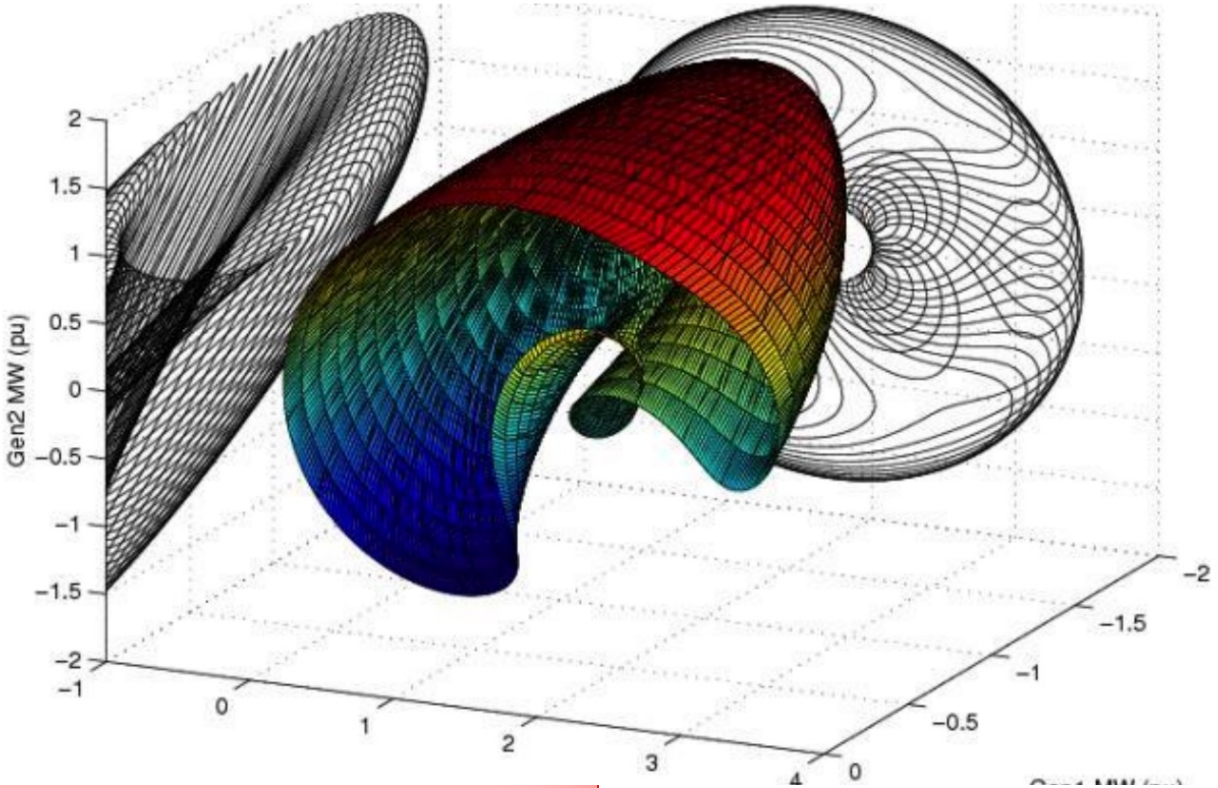


Fig. 6. Three bus system.



A knowledge of the solution boundary of the power flow problem is important for determining the robustness of operating points, and for evaluating strategies for improving robustness. A method of exploring that solution boundary has been developed.

Examples have demonstrated some of the possible forms that the solution boundary can exhibit. It appears that quite complicated behavior is possible. This could have a significant influence on the formulation of algorithms for optimally improving system robustness. It remains to fully explore these issues.

Minimize $\sum_{i \in \mathcal{G}} F_i(\boldsymbol{P_i^g})$

with constraints:

$\forall$ branch km: $\boldsymbol{S_{km}} =$
$(G_{kk} - jB_{kk})\, |\boldsymbol{V_k}|^2 + (G_{km} - jB_{km})\, |\boldsymbol{V_k}||\boldsymbol{V_m}|(\cos \boldsymbol{\theta_{km}} + \boldsymbol{j} \sin \boldsymbol{\theta_{km}})$

$$\sum_{km \in \delta(k)} \boldsymbol{S_{km}} = (\sum_{i \in \mathcal{G}(k)} \boldsymbol{P_i^g} - P_k^d) + j(\sum_{i \in \mathcal{G}(k)} \boldsymbol{Q_i^g} - Q_k^d)$$

Power flow limit on branch km:
$|\boldsymbol{S_{km}}|^2 = \mathcal{R}\mathrm{e}(\boldsymbol{S_{km}})^2 + Im(\boldsymbol{S_{km}})^2 \leq U_{km}$

Voltage limit on bus k:
$V_k^{\min} \leq |\boldsymbol{V_k}| \leq V_k^{\max}$

Generator output limits:
$P_i^{\min} \leq \boldsymbol{P_i^g} \leq P_i^{\max}$

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A common simplification: the DC approximation

$\forall$ branch km: $\boldsymbol{S_{km}} =$
$(G_{kk} - jB_{kk})\, |\boldsymbol{V_k}|^2 + (G_{km} - jB_{km})\, |\boldsymbol{V_k}||\boldsymbol{V_m}|(\cos \boldsymbol{\theta_{km}} + \boldsymbol{j} \sin \boldsymbol{\theta_{km}})$

- $|\boldsymbol{V_k}| = \mathbf{1}$ for all buses $\boldsymbol{k}$. Why?
- $\cos \boldsymbol{\theta_{km}} = \mathbf{1}$ and $\sin \boldsymbol{\theta_{km}} = \boldsymbol{\theta_k} - \boldsymbol{\theta_m}$ for all branches $\boldsymbol{km}$.
- Ignore reactive power.

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DC approximation

Minimize $\sum_{i \in \mathcal{G}} F_i(\boldsymbol{P_i^g})$

with constraints:

$\forall$ branch km: $\boldsymbol{P_{km}} = y_{km}(\boldsymbol{\theta_k} - \boldsymbol{\theta_m})$

$\sum_{km \in \delta(k)} \boldsymbol{P_{km}} = (\sum_{i \in \mathcal{G}(k)} \boldsymbol{P_i^g} - P_k^d)$

Power flow limit on branch km:
$|\boldsymbol{P_{km}}| \leq U_{km}$

Generator output limit on bus k:
$P_k^{\min} \leq \boldsymbol{P_k^g} \leq P_k^{\max}$

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Unit commitment (network aware, 1 period)

Minimize $\sum_{i \in \mathcal{G}} (F_i(\boldsymbol{P_i^g}) + h_i z_i)$

with constraints:

$\forall$ branch km: $\boldsymbol{P_{km}} = y_{km}(\boldsymbol{\theta_k} - \boldsymbol{\theta_m})$

$$\sum_{km \in \delta(k)} \boldsymbol{P_{km}} = (\sum_{i \in \mathcal{G}(k)} \boldsymbol{P_i^g} - P_k^d)$$

Power flow limit on branch km: $|\boldsymbol{P_{km}}| \leq U_{km}$

Generator output limits: $\boldsymbol{z_i}\, P_i^{\min} \leq \boldsymbol{P_i^g} \leq \boldsymbol{z_i}\, P_i^{\max}, \quad \boldsymbol{z_i} \in \{0, 1\}.$

Multi-period model has ramping constraints, plus more

Network operations

- Once a day: day-ahead markets

  Plans hourly operations for the next day, based on demand estimates and generator bids.
  Multi-period unit-commitment problem.
  Typically "security constrained". Also known as SCUC (security-constrained unit commitment)
  After solving, fix binaries, rerun as LP. Duals used as market prices.

- As the day progresses, more restricted/accurate versions of SCUC are run so as to correct

- Near real-time (5 minutes), DC OPF, also known as RT-dispatch. Duals used as spot prices.

What else:

- Real-time balancing provided by "reserves" setup. Paid through a different market. Planning also uses LP.

- Long-term planning using "capacity markets". Also LP/MIP.

- Today, AC models are only used for studies or to "make sure" that e.g. RT-dispatch is safe.

How do we "solve" ACOPF ?

- A tale of two worlds:
- **Good solutions** only or **lower bounds**, also?
- How about extensions to mixed-integer variants?

Good solutions: interior point methods

- A must-have tool! *(today)*
- Knitro, IPOPT. LOQO?
- Knitro and IPOPT are excellent.
- Even casual implementations obtain good results. No other algorithmic approach comes close.
- Also, very elegant theory!
- Theory only guarantees convergence to (?) a critical point for the barrier function.
- On standard ACOPF near-optimal solutions are routinely obtained, even on large cases.
- Polar formulation (not QCQP) is by far better.
- A (minor?) issue: solutions can exhibit small infeasibilities.

Sample runtimes

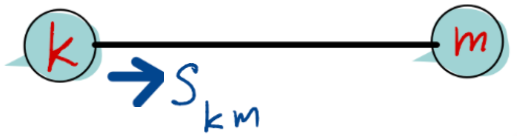
Case	# buses	# branches	# generators	Runtime (s)	Solver
118	118	186	54	0.76	MIPS
1354pegase	1354	1991	260	1.95	MIPS
ACTIVSg2000	2000	3206	544	2.96	MIPS
3120sp	3120	3693	505	4.25	MIPS
9241pegase	9241	16049	1445	11.78	MIPS
ACTIVSg70k	70000	88287	10390	~8 minutes	IPOPT, Knitro

But … how good are these solutions?

Convex relaxations of ACOPF

- Why do we care about lower bounds?
- Mixed-integer cases of ACOPF?
- Do we want an alternative to interior point algorithms for computing good solutions?

ACOPF as a **QCQP**
(Quadratically Constrained Quadratic Program)



complex power **injected** into **km** at **k**

$$V_k = |V_k| e^{j\theta_k}$$

$$V_m = |V_m| e^{j\theta_m}$$

$$\theta_{km} \doteq \theta_k - \theta_m$$

$$\boldsymbol{S_{km}} = (G_{kk} - jB_{kk})\,|\boldsymbol{V_k}|^2 + (G_{km} - jB_{km})\,|\boldsymbol{V_k}||\boldsymbol{V_m}|(\cos\boldsymbol{\theta_{km}} + \boldsymbol{j}\sin\boldsymbol{\theta_{km}})$$

$$S_{km} = V_k \left[Y \begin{pmatrix} V_k \\ V_m \end{pmatrix} \right]^*$$

$$Y = \begin{pmatrix} G_{kk} + jB_{kk} & G_{km} + jB_{km} \\ G_{mk} + jB_{mk} & G_{mm} + jB_{mm} \end{pmatrix}$$

Admittance matrix for line km

$$V_i = e_i + j f_i$$

Use rectangular coordinates for voltages

ACOPF as a **QCQP**

$$\boldsymbol{S_{km}} = (G_{kk} - jB_{kk})\, |\boldsymbol{V_k}|^2 + (G_{km} - jB_{km})\, |\boldsymbol{V_k}||\boldsymbol{V_m}|(\cos\boldsymbol{\theta_{km}} + \boldsymbol{j}\sin\boldsymbol{\theta_{km}})$$

$$V_i = e_i + j f_i$$

Use rectangular coordinates for voltages

Real ("active") part

$$\boldsymbol{P_{km}} = G_{kk}(\boldsymbol{e_k^2} + \boldsymbol{f_k^2}) + G_{km}(\boldsymbol{e_k e_m} + \boldsymbol{f_k e_m}) + B_{km}(-\boldsymbol{e_k f_m} + \boldsymbol{f_k e_m})$$

Imaginary ("reactive") part

$$\boldsymbol{Q_{km}} = -B_{kk}(\boldsymbol{e_k^2} + \boldsymbol{f_k^2}) - B_{km}(\boldsymbol{e_k e_m} + \boldsymbol{f_k e_m}) + G_{km}(-\boldsymbol{e_k f_m} + \boldsymbol{f_k e_m})$$

McCormick relaxation - an important workhorse

$w = xy$

$x^L \leq x \leq x^U \quad y^L \leq y \leq y^U$ where x^L, x^U, y^L, y^U are upper and lower bound values for x and y respectively

Convex hull provided by under/over estimators

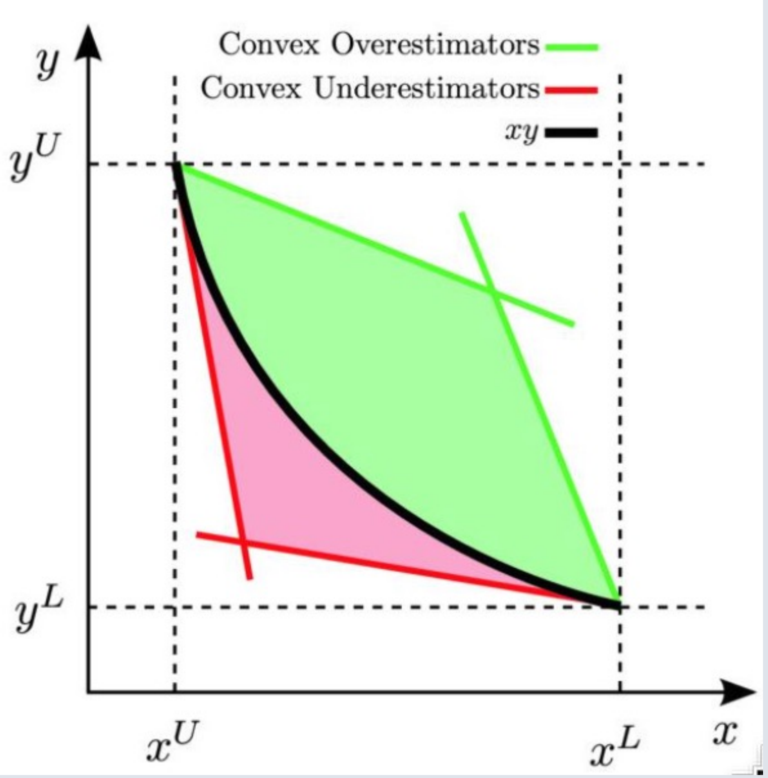
The underestimators of the function are represented by:

$$w \geq x^L y + x y^L - x^L y^L \; ; w \geq x^U y + x y^U - x^U y^U$$

The overestimators of the function are represented by:

$$w \leq x^U y + x y^L - x^U y^L \; ; w \leq x y^U + x^L y - x^L y^U$$

Works well in tandem with *spatial branching*



(source: Wikipedia)

And how well does it work?

Case	Root relaxation	300 seconds	Interior Point
9	2264.30	5301.40*	5296.69
30	0.00	154.08	576.89
118	0.00	0.00	129660.69
1354pegase	23037.69	23037.69	74069.35
ACTIVSg2000	649917.91	649917.91	1228892.08

(Gurobi 10 on QCQP)

Why so bad?

How about upper bounds?

A critical observation

$$\boldsymbol{S_{km}} = (G_{kk} - jB_{kk})\ |\boldsymbol{V_k}|^2 + (G_{km} - jB_{km})\ |\boldsymbol{V_k}||\boldsymbol{V_m}|(\cos\boldsymbol{\theta_{km}} + \boldsymbol{j}\sin\boldsymbol{\theta_{km}})$$

Real ("active") part

$$\boldsymbol{P_{km}} = G_{kk}\ |\boldsymbol{V_k}|^2 + G_{km}\ |\boldsymbol{V_k}||\boldsymbol{V_m}|\cos\boldsymbol{\theta_{km}} + B_{km}\ |\boldsymbol{V_k}||\boldsymbol{V_m}|\sin\boldsymbol{\theta_{km}}$$

Introduce new variables: $\boldsymbol{v_k^{(2)}} = |\boldsymbol{V_k}|^2$, $\boldsymbol{c_{km}} = |\boldsymbol{V_k}||\boldsymbol{V_m}|\cos\boldsymbol{\theta_{km}}$, $\boldsymbol{s_{km}} = |\boldsymbol{V_k}||\boldsymbol{V_m}|\sin\boldsymbol{\theta_{km}}$

$$\boldsymbol{P_{km}} = G_{kk}\ \boldsymbol{v_k^{(2)}} + G_{km}\ \boldsymbol{c_{km}} + B_{km}\ \boldsymbol{s_{km}}$$

And!

$$\boldsymbol{c_{km}^2} + \boldsymbol{s_{km}^2} \leq \boldsymbol{v_k^{(2)}}\boldsymbol{v_m^{(2)}}$$

Jabr inequality

New variables can be related to rectangular coordinates for voltages

$$\boldsymbol{P_{km}} = G_{kk}(\ \boldsymbol{e_k^2} + \boldsymbol{f_k^2}) + G_{km}(\ \boldsymbol{e_k}\boldsymbol{e_m} + \boldsymbol{f_k}\boldsymbol{e_m}) + B_{km}(\ -\boldsymbol{e_k}\boldsymbol{f_m} + \boldsymbol{f_k}\boldsymbol{e_m})$$

Minimize $\sum_{i\in\mathcal{G}} F_i(\ \boldsymbol{P_i^g})$

with constraints:

$$\sum_{km\in\delta(k)} \boldsymbol{S_{km}} \ = \ (\sum_{i\in\mathcal{G}(k)} \boldsymbol{P_i^g} - P_k^d) + j(\sum_{i\in\mathcal{G}(k)} \boldsymbol{Q_i^g} - Q_k^d)$$

$$\forall \text{ branch } km : \quad \boldsymbol{S_{km}} = \boldsymbol{P_{km}} + \boldsymbol{jQ_{km}}$$

$$\boldsymbol{P_{km}} = G_{kk}\ \boldsymbol{v_k^{(2)}} + G_{km}\ \boldsymbol{c_{km}} + B_{km}\ \boldsymbol{s_{km}}$$

$$\boldsymbol{Q_{km}} = -B_{kk}\ \boldsymbol{v_k^{(2)}} - B_{km}\ \boldsymbol{c_{km}} + G_{km}\ \boldsymbol{s_{km}}$$

$$\rightarrow \boldsymbol{v_k^{(2)}} = \boldsymbol{e_k^2} + \boldsymbol{f_k^2},\ \boldsymbol{c_{km}} = \boldsymbol{e_k e_m} + \boldsymbol{f_k f_m},\ \boldsymbol{s_{km}} = -\boldsymbol{e_k f_m} + \boldsymbol{f_k e_m}$$

$$\boldsymbol{P_{km}}^2 + \ \boldsymbol{Q_{km}}^2 \ \leq \ U_{km}$$

$$V_k^{\min} \leq \boldsymbol{v_k^{(2)}} \leq V_k^{\max} \quad \forall \text{ bus } k$$

$$P_k^{\min} \leq \boldsymbol{P_k^g} \leq P_k^{\max} \quad \forall \text{ generator } k$$

Minimize $\sum_{i\in\mathcal{G}} F_i(\boldsymbol{P_i^g})$

with constraints:

$$\sum_{km\in\delta(k)} \boldsymbol{S_{km}} = (\sum_{i\in\mathcal{G}(k)} \boldsymbol{P_i^g} - P_k^d) + j(\sum_{i\in\mathcal{G}(k)} \boldsymbol{Q_i^g} - Q_k^d)$$

$\forall$ branch km : $\boldsymbol{S_{km}} = \boldsymbol{P_{km}} + \boldsymbol{jQ_{km}}$

$$\boldsymbol{P_{km}} = G_{kk}\, \boldsymbol{v_k^{(2)}} + G_{km}\, \boldsymbol{c_{km}} + B_{km}\, \boldsymbol{s_{km}}$$

$$\boldsymbol{Q_{km}} = -B_{kk}\, \boldsymbol{v_k^{(2)}} - B_{km}\, \boldsymbol{c_{km}} + G_{km}\, \boldsymbol{s_{km}}$$

$$\rightarrow\ \boldsymbol{c_{km}^2} + \boldsymbol{s_{km}^2} \leq \boldsymbol{v_k^{(2)}} \boldsymbol{v_m^{(2)}}$$

$$\boldsymbol{P_{km}}^2 + \boldsymbol{Q_{km}}^2 \leq U_{km}$$

$V_k^{\min} \leq \boldsymbol{v_k^{(2)}} \leq V_k^{\max}$ $\forall$ bus k

$P_i^{\min} \leq \boldsymbol{P_i^g} \leq P_i^{\max}$ $\forall$ generator i

We can see an improvement

Case	Jabr relaxation	Relaxation time (s)	Log barrier	Interior point time (s)
9	5296.67	0.00	5296.69	0.24
30	573.58	0.03	576.89	0.47
118	129297.41	0.32	129660.69	0.24
1354pegase	740092.83	2.02	74069.35	2.45
ACTIVSg2000	1226328.77	4.29	1228892.08	3.01
3120sp	2130950.72	53.01	2142703.77	5.24
9241pegase	309238.37	31.00	315912.43	161.29
Larger examples		Non-convergence!		
9241pegase Non-convex	**84371.82**	**Root time: 400 s**		

(Gurobi 10 on SOCP)

Why?

Minimize $\sum_{i \in \mathcal{G}} F_i(\boldsymbol{P_i^g})$

with constraints:

$$\sum_{km \in \delta(k)} \boldsymbol{S_{km}} \;=\; (\sum_{i \in \mathcal{G}(k)} \boldsymbol{P_i^g} - P_k^d) \;+\; j(\sum_{i \in \mathcal{G}(k)} \boldsymbol{Q_i^g} - Q_k^d)$$

$\forall$ branch km : $\quad \boldsymbol{S_{km}} = \boldsymbol{P_{km}} + \boldsymbol{jQ_{km}}$

$$\boldsymbol{P_{km}} = G_{kk}\, \boldsymbol{v_k^{(2)}} + G_{km}\, \boldsymbol{c_{km}} + B_{km}\, \boldsymbol{s_{km}}$$

$$\boldsymbol{Q_{km}} = -B_{kk}\, \boldsymbol{v_k^{(2)}} - B_{km}\, \boldsymbol{c_{km}} - G_{km}\, \boldsymbol{s_{km}}$$

$$\rightarrow \quad \boldsymbol{c_{km}^2} + \boldsymbol{s_{km}^2} \;\leq\; \boldsymbol{v_k^{(2)}} \boldsymbol{v_m^{(2)}}$$

$$\boldsymbol{P_{km}}^2 + \boldsymbol{Q_{km}}^2 \;\leq\; U_{km}$$

$V_k^{\min} \leq \boldsymbol{v_k^{(2)}} \leq V_k^{\max} \quad \forall$ bus k

$P_i^{\min} \leq \boldsymbol{P_i^g} \leq P_i^{\max} \quad \forall$ generator i

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Another critical observation

$$S_{km} = (G_{kk} - jB_{kk})\,|V_k|^2 + (G_{km} - jB_{km})\,|V_k||V_m|(\cos\theta_{km} + j\sin\theta_{km})$$

Real ("active") part: $$P_{km} = G_{kk}\,|V_k|^2 + G_{km}\,|V_k||V_m|\cos\theta_{km} + B_{km}\,|V_k||V_m|\sin\theta_{km}$$

$$P_{km} = G_{kk}\,v_k^{(2)} + G_{km}\,c_{km} + B_{km}\,s_{km}$$

Observation:

(+ G. Munoz, 2014)

$$P_{km} + P_{mk} = \text{power “loss”} \geq 0$$

Using this inequality, and **foregoing the (rotated cone) Jabr ineq**., already yields a **very tight relaxation (linear?)**

Minimize $\sum_{i \in \mathcal{G}} F_i(\boldsymbol{P_i^g})$

with constraints:

$$\sum_{km \in \delta(k)} \boldsymbol{S_{km}} = (\sum_{i \in \mathcal{G}(k)} \boldsymbol{P_i^g} - P_k^d) + j(\sum_{i \in \mathcal{G}(k)} \boldsymbol{Q_i^g} - Q_k^d)$$

$\forall$ branch km : $\boldsymbol{S_{km}} = \boldsymbol{P_{km}} + \boldsymbol{jQ_{km}}$

$$\boldsymbol{P_{km}} = G_{kk}\, \boldsymbol{v_k^{(2)}} + G_{km}\, \boldsymbol{c_{km}} + B_{km}\, \boldsymbol{s_{km}}$$

$$\boldsymbol{Q_{km}} = -B_{kk}\, \boldsymbol{v_k^{(2)}} - B_{km}\, \boldsymbol{c_{km}} + G_{km}\, \boldsymbol{s_{km}}$$

$$\boldsymbol{P_{km}} + \boldsymbol{P_{mk}} \geq 0 \quad \leftarrow$$

$$\boldsymbol{P_{km}}^2 + \boldsymbol{Q_{km}}^2 \leq U_{km}$$

$V_k^{\min} \leq \boldsymbol{v_k^{(2)}} \leq V_k^{\max}$ $\quad \forall$ bus k

$P_i^{\min} \leq \boldsymbol{P_i^g} \leq P_i^{\max}$ $\quad \forall$ generator i

GUROBI OPTIMIZATION

The almost linear formulation (gray = conic)

Case	relaxation	Relax time (s)	Interior Point	IPM time (s)
1354pegase	730269.92 740092.83	0.81 2.02	74069.35	2.45
ACTIVSg2000	1201333.04 1226328.77	1.93 4.29	1228892.08	3.01
3120sp	2061763.91 2130950.72	2.60 53.01	2142703.77	5.24
3375wp	7267498.66 7393015.73	2.00 5.00	7412072.19	5.66
6468rte	84117.84 unable to converge	2.29 48.00	unable to converge	long
9241pegase	304392.01 309238.37	7.32 31.00	315912.43	161.29
ACTIVSg10k	2364513.20 unable to converge	7.23 **146.94**	2485898.75	129.54
13569pegase	372046.88 unable to converge	6.98 76.00	3861075.19	320.00
ACTIVSg25k	5802299.18 unable to converge	125 **332**	6017830.61	86.07
ACTIVSg70k	15305723.20 unable to converge	162.10 684.05	16439499.87	450.55

$$S_{km} = (G_{kk} - jB_{kk})\,|V_k|^2 + (G_{km} - jB_{km})\,|V_k||V_m|(\cos\theta_{km} + j\sin\theta_{km})$$

$$G_{kk} > 0 > G_{km} = G_{mk} \geq -G_{kk}, \quad B_{km} = B_{mk}$$

$$P_{km} = G_{kk}\,|V_k|^2 + G_{km}\,|V_k||V_m|\cos\theta_{km} + B_{km}\,|V_k||V_m|\sin\theta_{km}$$

$$P_{km} = G_{kk}\,v_k^{(2)} + G_{km}\,c_{km} + B_{km}\,s_{km}$$

$$P_{mk} = G_{mm}\,v_m^{(2)} + G_{mk}\,c_{km} - B_{km}\,s_{km}$$

$$P_{km} + P_{mk} = G_{kk}\,v_k^{(2)} + G_{mm}\,v_m^{(2)} + 2G_{km}\,c_{km} \geq \min\{G_{kk}, G_{mm}\}(\,v_k^{(2)} + v_m^{(2)} - 2c_{km})$$

$$c_{km}^2 + s_{km}^2 \leq v_k^{(2)} v_m^{(2)}$$

$$v_k^{(2)} + v_m^{(2)} - 2c_{km} \geq 0$$

$$P_{km} + P_{mk} \geq 0$$

Lemma: this implies flow decomposition with source-destination flow paths

Other convex relaxations

- SDP (Semidefinite programming)
  The tightest relaxation, but impracticable

- OBBT
  In the literature, applied to the SOCP formulation. Too expensive. LP-based?

- Approximations to sine, cosine, arctangent.
  Only valid when

$$|\theta_k - \theta_m| \leq \pi/2$$

  Some of the inequalities are linear and some are convex quadratics or rotated cones

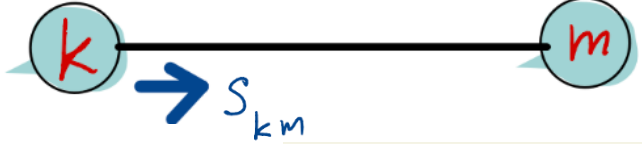
- It has been claimed that these formulations lead to a better starting point for IPOPT

- In our opinion, the Jabr inequality (or equivalent) does most of the work

- But the convex nonlinear relaxations are too expensive. Linear is better.

Moving forward

- **The nonlinear formulation will become more widely used**
- **Topology optimization**
  Binary variables used to turn off branches (or buses).
- **Configurable components**
- **The ongoing GO (Grid Optimization) competitions have had ~~killer~~ interesting features**



$$Y = \begin{pmatrix} G_{kk} + jB_{kk} & G_{km} + jB_{km} \\ G_{mk} + jB_{mk} & G_{mm} + jB_{mm} \end{pmatrix}$$

Admittance matrix for line km

$$I_{km} = Y \begin{pmatrix} V_k \\ V_m \end{pmatrix} \rightarrow S_{km} = V_k I_{km}^*$$

Ohm's Law

Complex current

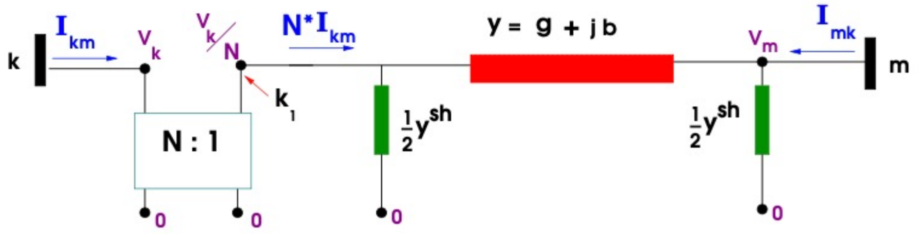
$$S_{km} = V_k \left[Y \begin{pmatrix} V_k \\ V_m \end{pmatrix} \right]^*$$

$$V_k = |V_k| e^{j\theta_k} \qquad \theta_{km} \doteq \theta_k - \theta_m \qquad V_m = |V_m| e^{j\theta_m}$$

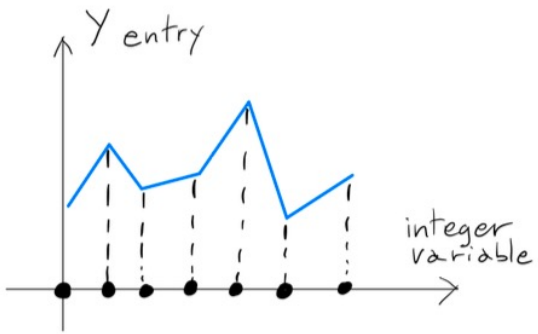
$$\boldsymbol{S_{km}} = (G_{kk} - jB_{kk})\, |\boldsymbol{V_k}|^2 + (G_{km} - jB_{km})\, |\boldsymbol{V_k}||\boldsymbol{V_m}|(\cos \boldsymbol{\theta_{km}} + \boldsymbol{j} \sin \boldsymbol{\theta_{km}})$$

GO competition: configurable transformers and shunts

- Example: tap ratio and angle in a transformer can be adjusted



- Impedance correction factors modeled using a piecewise-linear curve



- Function can be quite nonlinear with local optima

- Switched shunts, in blocks (at buses)

- Altogether, a large number of integer variables

GUROBI OPTIMIZATION

Let's get started: topology optimization on standard ACOPF

Minimize $\sum_{i \in \mathcal{G}} F_i(\,\boldsymbol{P_i^g})$

with constraints:

$\forall$ branch km:

$$\boldsymbol{S_{km}} = \boldsymbol{z_{km}} \left[\,(G_{kk} - jB_{kk})\,|\boldsymbol{V_k}|^2 + (G_{km} - jB_{km})\,|\boldsymbol{V_k}||\boldsymbol{V_m}|(\cos\boldsymbol{\theta_{km}} + \boldsymbol{j}\sin\boldsymbol{\theta_{km}})\,\right]$$

$$\boldsymbol{z_{km}} = \boldsymbol{0\ or\ 1}$$

$$\sum_{km \in \delta(k)} \boldsymbol{S_{km}} = (\sum_{i \in \mathcal{G}(k)} \boldsymbol{P_i^g} - P_k^d) + j(\sum_{i \in \mathcal{G}(k)} \boldsymbol{Q_i^g} - Q_k^d)$$

Power flow limit on branch km:

$$|\boldsymbol{S_{km}}|^2 = \mathcal{R}\mathrm{e}(\,\boldsymbol{S_{km}})^2 + Im(\,\boldsymbol{S_{km}})^2 \leq U_{km}$$

Voltage limit on bus k:

$$V_k^{\min} \leq |\boldsymbol{V_k}| \leq V_k^{\max}$$

Generator output limits:

$$P_i^{\min} \leq \boldsymbol{P_i^g} \leq P_i^{\max}$$

GUROBI OPTIMIZATION

Convex relaxation + binary variables

Minimize $\sum_{i \in \mathcal{G}} F_i(\boldsymbol{P_i^g})$

with constraints:

$$\sum_{km \in \delta(k)} \boldsymbol{S_{km}} = (\sum_{i \in \mathcal{G}(k)} \boldsymbol{P_i^g} - P_k^d) + j(\sum_{i \in \mathcal{G}(k)} \boldsymbol{Q_i^g} - Q_k^d)$$

$$\forall \text{ branch } km : \quad \boldsymbol{S_{km}} = \boldsymbol{z_{km}} [\, \boldsymbol{P_{km}} + \boldsymbol{jQ_{km}} \,], \quad \boldsymbol{z_{km}} \in \{0, 1\}$$

$$\boldsymbol{P_{km}} = G_{kk}\, \boldsymbol{v_k^{(2)}} + G_{km}\, \boldsymbol{c_{km}} + B_{km}\, \boldsymbol{s_{km}}$$

$$\boldsymbol{Q_{km}} = -B_{kk}\, \boldsymbol{v_k^{(2)}} - B_{km}\, \boldsymbol{c_{km}} + G_{km}\, \boldsymbol{s_{km}}$$

$$\boldsymbol{c_{km}^2} + \boldsymbol{s_{km}^2} \leq \boldsymbol{v_k^{(2)}} \boldsymbol{v_m^{(2)}}$$

$$\boldsymbol{P_{km}}^2 + \boldsymbol{Q_{km}}^2 \leq U_{km}$$

$$V_k^{\min} \leq \boldsymbol{v_k^{(2)}} \leq V_k^{\max} \quad \forall \text{ bus } k$$

$$P_i^{\min} \leq \boldsymbol{P_i^g} \leq P_i^{\max} \quad \forall \text{ generator } i$$

GUROBI OPTIMIZATION

A mixed-integer set in topology optimization

Binary variables $\boldsymbol{z_{km}}$

$$\begin{aligned} \boldsymbol{z_{km}} &= 1, \quad \text{if branch } \mathbf{km} \text{ is turned on,} \\ &= 0, \quad \text{otherwise.} \end{aligned}$$

$$\boldsymbol{z_{km}} = 1 \rightarrow \begin{aligned} \boldsymbol{P_{km}} &= G_{kk}\, \boldsymbol{v_k^{(2)}} + G_{km}\, \boldsymbol{c_{km}} + B_{km}\, \boldsymbol{s_{km}} \\ \boldsymbol{P_{mk}} &= G_{mm}\, \boldsymbol{v_m^{(2)}} + G_{mk}\, \boldsymbol{c_{km}} - B_{km}\, \boldsymbol{s_{km}} \end{aligned}$$

$$\boldsymbol{z_{km}} = 0 \rightarrow \boldsymbol{P_{km}} = \boldsymbol{P_{mk}} = \boldsymbol{c_{km}} = \boldsymbol{s_{km}} = 0$$

Or $\boldsymbol{P_{km}} = \boldsymbol{P_{mk}} = 0$

And in both cases: $$\boldsymbol{v_k^{(2)} + v_m^{(2)} - 2c_{km} \geq 0}$$

$$\rightarrow \boldsymbol{P_{km} + P_{mk} \geq 0}$$

ACOPF as Optimod

Soon to be rolled out

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QUESTIONS?

Thank You

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